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M338 Topology

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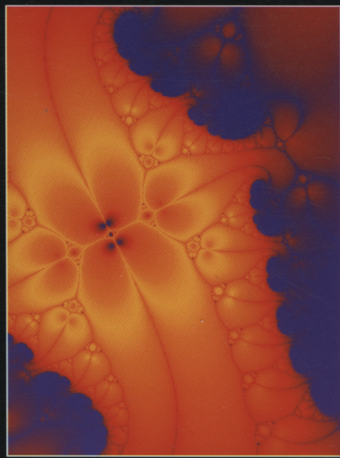
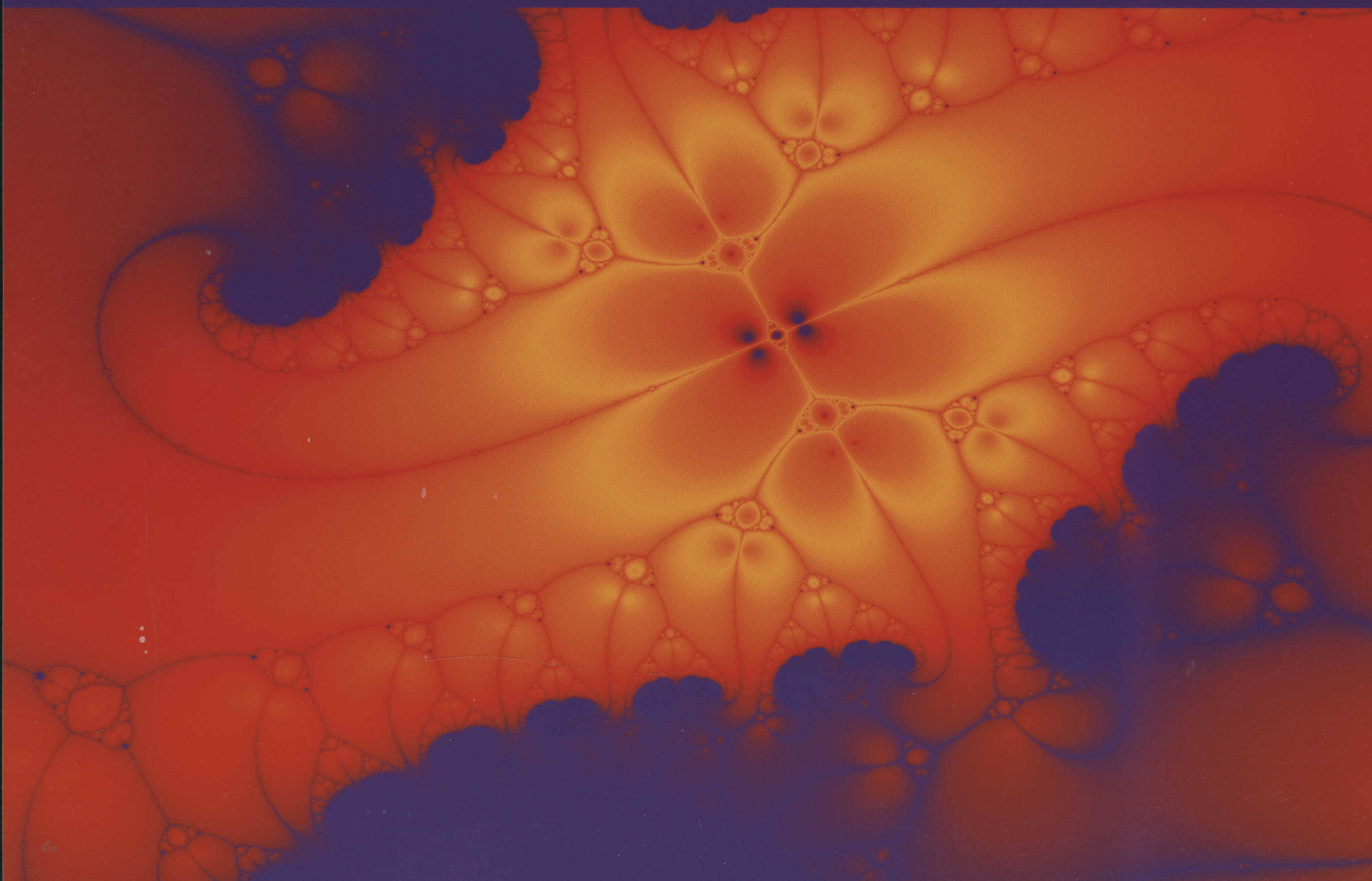


Unit A3 Topological spaces



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M338 Topology

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Topological spaces

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1.1

Contents

Introduction

We began this block by studying continuous real functions and introducing the ε–δ definition of continuity. We then explained how this definition can be generalized to include continuous functions between metric spaces. At the end of *Unit A2* we showed that continuity for functions between metric spaces can be defined in terms of open sets. This means that we no longer need a notion of distance for continuity to be defined, but simply a collection of open sets satisfying certain properties.

In this unit, we specify the properties that a collection of open subsets of a set X should possess so that we can use the open set definition of continuity. We call such a collection of subsets of X a *topology* on X. The set X, together with a topology on X, is then known as a *topological space*. We show that each metric on a set determines a topological space. We also give some examples of topological spaces that are *not* metric spaces; these show that the concept of a topological space is more general than that of a metric space. This is the culmination of our search for the most general setting in which continuity can be defined.

Study guide

The most important sections of this unit are Sections 1–4, and you should expect to spend most of your study time on these sections.

In Section 1, *Introducing topological spaces*, we define a *topological space* and *continuity* on a topological space, and give some simple examples of topological spaces.

Section 2, *Sets and families*, introduces some notation and results from set theory. This enables us, in Section 3, *Examples of topological spaces*, to introduce some interesting and challenging examples, including those that arise as *subspaces* of other topological spaces.

In Section 4, *Continuity for topological spaces*, we look at some important examples of continuous functions between topological spaces.

Section 5, *Bases*, introduces the idea of a *base* for a topology — a collection of open sets that form the building blocks for the topology.

Finally, in Section 6, *Comparing topologies*, we examine the consequences of having two topologies on the same set, one with more open sets than the other.

This unit uses many results from set theory. Some of these appear in Section 2 and others are listed in the Handbook. We do not expect you to prove most of these results, but you should be able to use them. You may wish to spend a short time familiarizing yourself with the set theory results in the Handbook before you continue.

There is no software associated with this unit.

1 Introducing topological spaces

After working through this section, you should be able to:

- state and use the definition of a *topological space*;
- define the *discrete* and *indiscrete* topologies on a given set;
- explain the meaning of the term *metrizable*;
- define *continuity* for functions between general topological spaces.

We begin by revisiting the open set definition of continuity for functions between metric spaces, with the aim of obtaining a more general class of spaces for which this definition applies. This leads us to the definition of a *topological space* — a set X, together with a collection of open subsets of X satisfying certain properties.

You saw this approach to continuity in Section 4 of *Unit A2*.

We then give some simple examples of topological spaces and continuous functions between such spaces. Some of these examples may seem rather trivial, but the advantage of beginning with them is that they will help you to become familiar with the new terminology associated with topological spaces. You will meet some more interesting examples of topological spaces in Section 3.

1.1 What is a topological space?

In *Unit A2*, we showed how the ε–δ definition of continuity for real functions can be generalized to provide the following definition of continuity for functions between *any* two metric spaces X and Y.

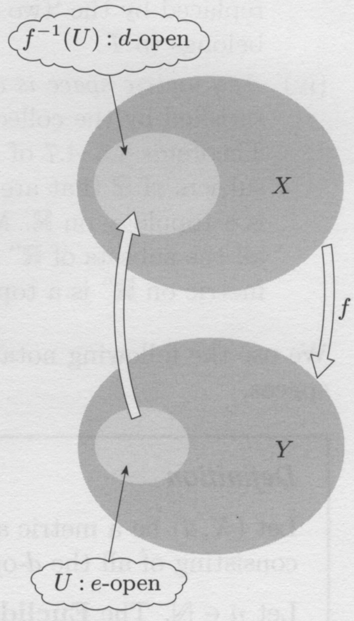

Figure 1.1

> ***Definition***
>
> Let (X,d) and (Y,e) be metric spaces.
>
> A function $f\colon X \to Y$ is **continuous** on X if $f^{-1}(U)$ is a d-open subset of X whenever U is an e-open subset of Y.

Unlike our first definition of continuity for functions between metric spaces (*Unit A2*, Section 1), this definition is not written in terms of distances between points; instead, it involves only the open sets in X and Y. This suggests that we may be able to generalize our definition of continuity still further, to functions between even more general spaces. Such spaces should have well-defined open sets, but they need not have any notion of a metric or distance associated with them.

In order to define such a space, we must establish what properties the open sets should have. In *Unit A2*, we investigated the properties of open sets in metric spaces and established the following results.

Unit A2, Theorems 4.5–4.7.

> - The empty set and the whole space are open sets.
> - The intersection of any *two* open sets is an open set.
> - The union of *any* collection of open sets is an open set.

It turns out that these are the *only* properties we require of open sets. A set X, together with a collection $\mathcal{T}$ of subsets of X satisfying these three conditions, is known as a *topological space*.

> ***Definition***
>
> Let X be a set and let $\mathcal{T}$ be a collection of subsets of X.
> $\mathcal{T}$ is a **topology** on X if the following three conditions are satisfied:
>
> (T1) the sets $\varnothing$ and X belong to $\mathcal{T}$;
>
> (T2) the intersection of any *two* sets in $\mathcal{T}$ belongs to $\mathcal{T}$;
>
> (T3) the union of *any* collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.
>
> The set X, together with the collection $\mathcal{T}$, is a **topological space**, written $(X, \mathcal{T})$. The sets $U \in \mathcal{T}$ are the **open** sets in X.

Remarks

(i) If we wish to emphasize the topology $\mathcal{T}$, then we refer to U as $\mathcal{T}$*-open*, or *open for* $\mathcal{T}$, or *open in* $(X, \mathcal{T})$ or *open with respect to* $\mathcal{T}$.

(ii) Arguing as in *Unit A2*, Section 4, we may conclude from (T2) that the intersection of any *finite* collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$. Indeed, many books on topology replace (T2) with the condition that the intersection of any finite collection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$. We prefer the 'two sets' form of the condition because it is easier to use in practice.

Also arguing as in *Unit A2*, Section 4, we can conclude that the intersection of *infinitely* many open sets need not be open.

(iii) For topologies where $\mathcal{T}$ is a *finite* collection of sets, (T3) may also be replaced by the 'two sets' form: the union of any *two* sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

(iv) *Any metric space is a topological space*, since (T1)–(T3) are properties satisfied by the collection of open sets in a metric space (by Theorems 4.5–4.7 of *Unit A2*). For example, the collection of all the subsets of $\mathbb{R}$ that are open with respect to the Euclidean metric on $\mathbb{R}$ is a topology on $\mathbb{R}$. More generally, for each $n \in \mathbb{N}$, the collection of all the subsets of $\mathbb{R}^n$ that are open with respect to the Euclidean metric on $\mathbb{R}^n$ is a topology on $\mathbb{R}^n$.

We use the following notation and terminology in connection with metric spaces.

> ***Definition***
>
> Let (X, d) be a metric space. Then $\mathcal{T}(d)$ denotes the topology on X consisting of all the d-open subsets of X.
>
> Let $n \in \mathbb{N}$. The **Euclidean topology** on $\mathbb{R}^n$ is the topology $\mathcal{T}(d^{(n)})$, where $d^{(n)}$ is the Euclidean metric on $\mathbb{R}^n$.

In Section 6, we show that two different metrics can determine the same topology — that is, we can have $\mathcal{T}(d_1) = \mathcal{T}(d_2)$ even when $d_1 \neq d_2$.

Our aim in introducing the idea of a topological space is to obtain a more general class of spaces for which continuity can be defined. We provide that definition in Subsection 1.3. First, in Subsection 1.2, we look at some simple examples of topological spaces and we show that there are topologies that cannot be defined by metrics.

1.2 Simple examples of topological spaces

There are many collections of sets that satisfy (T1)–(T3). We begin by looking at some of the simplest of these.

First, it follows from (T1) that the smallest collection of sets that can possibly form a topology on a set X is $\{\varnothing, X\}$. We now show that this collection of sets does indeed form a topology on X.

Worked problem 1.1

Let X be a set. Show that $\mathcal{T} = \{\varnothing, X\}$ is a topology on X.

Solution

We must show that $\mathcal{T}$ satisfies (T1)–(T3).

(T1) $\varnothing$ and X both belong to $\mathcal{T}$, and so (T1) is satisfied.

(T2) For each set U in $\mathcal{T}$, $U \cap U = U \in \mathcal{T}$; so we only need to check intersections of *distinct* sets in $\mathcal{T}$. Thus, the only pair of sets that we need to consider is the pair $\varnothing, X$. Now $\varnothing \cap X = \varnothing$ and $\varnothing \in \mathcal{T}$, so (T2) is satisfied.

(T3) Since the union of any number of copies of a set U in $\mathcal{T}$ is simply U, we need only check unions of *distinct* sets in $\mathcal{T}$. Thus, again, we need consider only the pair $\varnothing, X$. Now $\varnothing \cup X = X$ and $X \in \mathcal{T}$, so (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}$ is a topology on X. ■

Remark

In general, when checking (T2) we need only check intersections of *distinct* sets in $\mathcal{T}$. This is because for all sets U, $U \cap U = U$.

Similarly, since the union of any number of copies of the same set is simply that set, when checking (T3) we need only check unions of *distinct* sets in $\mathcal{T}$.

At the other extreme, the *largest* collection of sets that can form a topology on a set X is the collection of *all* the subsets of X. We now ask you to show that this collection forms a topology on X.

Problem 1.1

Let X be a set. Show that the collection $\mathcal{T}$ of *all* the subsets of X satisfies (T1)–(T3) and is therefore a topology on X.

The two topologies you have just met have special names.

> ***Definition***
>
> Let X be any set.
>
> The **indiscrete topology** on X is the topology $\{\varnothing, X\}$.
>
> The **discrete topology** on X is the topology comprising all the subsets of X.

Remark

In *Unit A2* you met the *discrete metric* d_0 and saw (in Problem 4.6) that, for any set X, every subset of X is d_0-open. Thus the discrete topology on X is $\mathcal{T}(d_0)$, the collection of d_0-open subsets of X.

Problem 1.2

Let $X = \{a\}$ be a set with only one element. Show that the discrete and indiscrete topologies on X are the same, and that this is the only possible topology on X.

If a set $X = \{a, b\}$ has two elements, then the indiscrete topology on X is $\{\varnothing, X\}$ and the discrete topology on X is $\{\varnothing, \{a\}, \{b\}, X\}$. There are two further topologies on X:

$$\{\varnothing, \{a\}, X\} \quad \text{and} \quad \{\varnothing, \{b\}, X\}.$$

The number of topologies on a set X increases rapidly with the number of elements in the set X. There are, in fact, 29 possible topologies on a set with three elements.

No simple formula is known for the number of topologies on a set with n elements.

Worked problem 1.2

Let $X = \{a, b, c\}$. Which of the following are topologies on X?

(a) $\mathcal{T}_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$

(b) $\mathcal{T}_2 = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$

(c) $\mathcal{T}_3 = \{\varnothing, \{a\}, \{b\}, X\}$

Solution

In each case, we check whether (T1)–(T3) are satisfied. A good way to do this is to look first for obvious reasons why a condition is *not* satisfied (since finding such a reason will immediately establish that the collection is *not* a topology). If this initial search fails, then the next step is to look for reasons why each condition *is* satisfied.

(a) (T1) is not satisfied, since $X \notin \mathcal{T}_1$. Thus $\mathcal{T}_1$ is not a topology on X.

(b) There are no obvious reasons why (T1)–(T3) do not hold, so we try to establish each condition in turn.

(T1) The sets $\varnothing, X \in \mathcal{T}_2$, so (T1) is satisfied.

(T2) For any set U in $\mathcal{T}_2$, $\varnothing \cap U = \varnothing \in \mathcal{T}_2$ and $U \cap X = U \in \mathcal{T}_2$. So we need only check intersections not involving $\varnothing$ or X. We have $\{a\} \cap \{a, b\} = \{a\} \in \mathcal{T}_2$, $\{a\} \cap \{a, c\} = \{a\} \in \mathcal{T}_2$ and $\{a, b\} \cap \{a, c\} = \{a\} \in \mathcal{T}_2$. Thus all intersections of pairs of sets in $\mathcal{T}_2$ belong to $\mathcal{T}_2$, so (T2) is satisfied.

(T3) For any set U in $\mathcal{T}_2$, $\varnothing \cup U = U \in \mathcal{T}_2$ and $U \cup X = X \in \mathcal{T}_2$. So we need only check unions not involving $\varnothing$ or X. We have $\{a\} \cup \{a, b\} = \{a, b\} \in \mathcal{T}_2$, $\{a\} \cup \{a, c\} = \{a, c\} \in \mathcal{T}_2$ and $\{a, b\} \cup \{a, c\} = \{a, b, c\} = X \in \mathcal{T}_2$. Thus all unions of pairs of sets in $\mathcal{T}_2$ belong to $\mathcal{T}_2$, so (T3) is satisfied.

Recall from Remark (iii) following the definition of a topology that, for *finite* topologies, we may use the 'two sets' form of (T3).

Since (T1)–(T3) are satisfied, $\mathcal{T}_2$ is a topology on X.

(c) (T3) is not satisfied, since $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_3$. Thus $\mathcal{T}_3$ is not a topology on X. ■

Remark

Note that, once (T1) has been checked (so that $\varnothing$ and X belong to $\mathcal{T}$), we do *not* have to check intersections and unions involving $\varnothing$ or X (as indicated in the solution to (b)).

Problem 1.3

Let $X = \{a, b, c\}$. Which of the following are topologies on X?

(a) $\mathcal{T}_1 = \{\varnothing, \{a, b\}, \{a, c\}, X\}$

(b) $\mathcal{T}_2 = \{\{a\}, \{b\}, \{a, b\}, X\}$

(c) $\mathcal{T}_3 = \{\varnothing, \{b\}, \{a, b\}, X\}$

We have noted that all metric spaces are topological spaces. However, the converse is not true, as we shall show in Theorem 1.1. But first, we need a definition.

See Remark (iv) following the definition of a topology.

Definition

A topology $\mathcal{T}$ on a set X is **metrizable** if $\mathcal{T}$ is equal to $\mathcal{T}(d)$, for some metric d on X.

Recall that $\mathcal{T}(d)$ is our notation for the collection of d-open subsets of X.

Remark

Thus the discrete topology is metrizable as it is equal to $\mathcal{T}(d_0)$.

Here now is the promised result.

Theorem 1.1

Let X be a set with at least two elements. Then the indiscrete topology on X is not metrizable.

Proof Let $\mathcal{T}$ denote the indiscrete topology on X. Let d be any metric on the set X. This metric generates a topology $\mathcal{T}(d)$, and we need to show that $\mathcal{T}(d)$ cannot be the same as $\mathcal{T}$.

We know that the only sets in $\mathcal{T}$ are $\varnothing$ and X. Thus, to show that $\mathcal{T}$ is not metrizable, it is sufficient to find a set that belongs to $\mathcal{T}(d)$ but is neither $\varnothing$ nor X.

To do this, we take $a, b \in X$ with $a \neq b$ (so that $d(a, b) > 0$), and take $r > 0$ with $r < d(a, b)$. Then the ball $B_d(a, r)$ is d-open, and so belongs to $\mathcal{T}(d)$. However, $B_d(a, r) \neq \varnothing$ (since $a \in B_d(a, r)$) and $B_d(a, r) \neq X$ (since $b \notin B_d(a, r)$).

Thus, $\mathcal{T}$ cannot be $\mathcal{T}(d)$, no matter what choice we make of d, and so $\mathcal{T}$ is not metrizable. ∎

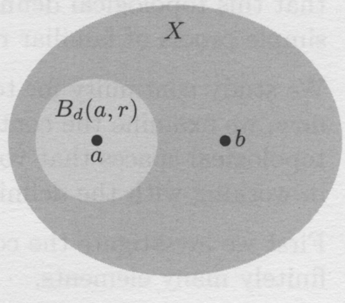

Figure 1.2

Theorem 1.1 shows that, for each set with at least two elements, there is at least one topological space that is not a metric space. In fact, though we shall not show this here, there are many topological spaces that are not metric spaces. So, as we have already suggested, the concept of a topological space is more general than that of a metric space.

1.3 Continuity in topological spaces

When we first introduced topological spaces, our aim was to obtain a more general class of spaces for which continuity can be defined. We saw in Subsection 1.2 that topological spaces do indeed form a more general class of spaces than metric spaces, and so we now turn to defining continuity on this more general class. We begin by rewriting our open set definition of continuity for *metric* spaces using the notation of topological spaces.

> Let (X, d) and (Y, e) be metric spaces.
>
> A function $f: X \to Y$ is *continuous* if $f^{-1}(U) \in \mathcal{T}(d)$ whenever $U \in \mathcal{T}(e)$.

This suggests the following definition of continuity for functions between *topological* spaces.

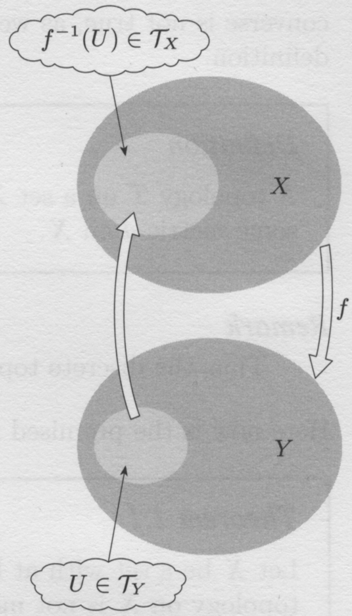

Figure 1.3

> ***Definition***
>
> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
>
> A function $f: X \to Y$ is **continuous** if $f^{-1}(U) \in \mathcal{T}_X$ whenever $U \in \mathcal{T}_Y$.

Remarks

(i) If we wish to emphasize the topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, then we say that f is $(\mathcal{T}_X, \mathcal{T}_Y)$*-continuous* or that f is *continuous from* $(X, \mathcal{T}_X)$ *to* $(Y, \mathcal{T}_Y)$.

(ii) If $\mathcal{T}_X$ and $\mathcal{T}_Y$ are metrizable using metrics d and e respectively, i.e. $\mathcal{T}_X = \mathcal{T}(d)$ and $\mathcal{T}_Y = \mathcal{T}(e)$, we can deduce that a function f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous if and only if it is (d, e)-continuous.

We now have a definition of continuity that can be used in a very general setting and covers all the particular cases that you met in earlier units. For example, in *Unit A1* you met several continuous real functions. We showed that these functions are continuous with respect to the Euclidean distance function (or metric) on $\mathbb{R}$. We can now say that these functions are continuous with respect to the Euclidean topology on $\mathbb{R}$. You will see later that this topological definition of continuity can provide particularly simple proofs of familiar results for real functions.

We study continuity for topological spaces in more detail in Section 4. For now, we examine the continuity of functions between some simple topological spaces that you have already met, in order to give you practice in working with the definition.

First we investigate the continuity of some functions between sets with finitely many elements.

Worked problem 1.3

Let $X = \{a, b, c\}$ and $Y = \{s, t\}$.
Let $f: X \to Y$ be defined by $f(a) = f(c) = s$ and $f(b) = t$.
Determine whether f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous in each of the following cases.

(a) $\mathcal{T}_X = \{\varnothing, \{b\}, \{a, b\}, X\}$, $\quad \mathcal{T}_Y = \{\varnothing, \{s\}, Y\}$.

(b) $\mathcal{T}_X = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\quad \mathcal{T}_Y = \{\varnothing, \{s\}, Y\}$.

You saw earlier that $\mathcal{T}_X$ and $\mathcal{T}_Y$ are topologies on X and Y, respectively.

Solution

We begin by noting that $f^{-1}(\{s\}) = \{a, c\}$ and $f^{-1}(\{t\}) = \{b\}$. The topology $\mathcal{T}_Y$ is the same in both cases, and the inverse images of the sets in $\mathcal{T}_Y$ are:

$$f^{-1}(\varnothing) = \varnothing, \quad f^{-1}(\{s\}) = \{a, c\}, \quad f^{-1}(Y) = f^{-1}(\{s, t\}) = \{a, b, c\} = X.$$

(a) The inverse image of $\{s\}$ does not belong to $\mathcal{T}_X$, and so f is not $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous.

(b) The inverse images of all the sets in $\mathcal{T}_Y$ belong to $\mathcal{T}_X$, and so f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous. ■

Problem 1.4

Let $X = \{a, b, c\}$ and $Y = \{r, s, t\}$.
Let $f: X \to Y$ be defined by $f(a) = s$ and $f(b) = f(c) = t$.
Determine whether f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous in each of the following cases.

(a) $\mathcal{T}_X = \{\varnothing, \{a\}, X\}, \quad \mathcal{T}_Y = \{\varnothing, \{s\}, \{r, s\}, Y\}$.

(b) $\mathcal{T}_X = \{\varnothing, \{b\}, X\}, \quad \mathcal{T}_Y = \{\varnothing, \{s\}, \{r, s\}, Y\}$.

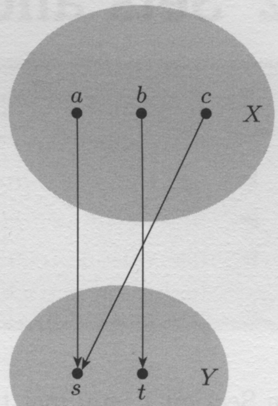

Figure 1.4

You may assume that $\mathcal{T}_X$ and $\mathcal{T}_Y$ are topologies on X and Y, respectively.

We end this section by considering continuity for the discrete and indiscrete topologies.

Worked problem 1.4

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and suppose that $\mathcal{T}_X$ is the discrete topology on X. Show that every function $f: X \to Y$ is continuous.

In *Unit A2*, Theorem 2.2, we saw that any function $f: X \to Y$ is continuous as a map between metric spaces, provided that the metric on X is the discrete metric.

Solution

Suppose that U belongs to the topology $\mathcal{T}_Y$ on Y. Then $f^{-1}(U)$ is a subset of X. Since *every* subset of X belongs to the discrete topology $\mathcal{T}_X$ on X, we have $f^{-1}(U) \in \mathcal{T}_X$. Thus, f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous. ■

Problem 1.5

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and suppose that $\mathcal{T}_Y$ is the indiscrete topology on Y. Show that every function $f: X \to Y$ is continuous.

We have thus proved the following result.

Theorem 1.2

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then every function $f: X \to Y$ is continuous if *either* $\mathcal{T}_X$ is the discrete topology on X *or* $\mathcal{T}_Y$ is the indiscrete topology on Y.

From this we deduce that changing the topologies on the domain or codomain can affect the continuity of a function. You will see more examples of this in Section 4.

In *Unit A2*, Subsection 2.1, we saw that changing the *metrics* on the domain or codomain can affect the continuity of a function.

2 Sets and families

After working through this section, you should be able to:

- distinguish between *finite*, *countably infinite* and *uncountable* sets;
- use the *index notation* for families of sets;
- define the *complement* of a set;
- use various results concerning the union and intersection of sets, including the *Distributive Laws* and *De Morgan's Laws*.

In Section 1, you met the discrete and indiscrete topologies and saw that any metric on a set determines a topology on that set. You also investigated some topologies on sets with only finitely many elements. In order to explore topologies on sets with *infinitely* many elements, it is necessary to understand the properties of intersections and unions of collections of infinitely many sets, and this is the aim of this section.

We begin, in Subsection 2.1, by introducing the idea of *countable* and *uncountable* sets, and give examples of these. Subsection 2.2 goes on to explain what are meant by a *family of sets* and by an *index set* for such a family. Finally, Subsection 2.3 contains the results about unions and intersections of families of sets that we need later.

2.1 Countable and uncountable sets

Recall that a *finite set* is one with finitely many elements, and that an *infinite set* is one that is not finite. There are two types of infinite sets that we wish to distinguish — those that are *countable* and those that are *uncountable.* Informally, we say that an infinite set is 'countable' if all of its elements can be written down as an infinite 'list', one after another, so that they can then be *counted.* An infinite set that cannot be listed in this way is said to be 'uncountable'.

For example, the elements of the set $\mathbb{N}$ of natural numbers can be listed in the following way:

$$1, 2, 3, \ldots;$$

so this set is countable (according to our informal definition). Similarly the set $\mathbb{Z}$ of integers can be listed as

$$0, 1, -1, 2, -2, \ldots$$

Every integer appears somewhere in the list.

and so this set too is countable (according to our informal definition).

Note that, if the elements of a set can be *listed*, then they can be *numbered.* For example, using our listing for the set $\mathbb{Z}$, we can say that 0 is the first element of $\mathbb{Z}$, 1 is the second element, -1 is the third element, and so on. Thus we can assign a natural number to each of them, as follows:

$$0 \mapsto 1,\ 1 \mapsto 2,\ -1 \mapsto 3,\ \ldots\ .$$

In effect, we have defined a one–one map from $\mathbb{Z}$ to $\mathbb{N}$.

However, it is not obvious whether we could list all the elements of the set $\mathbb{R}$ of real numbers.

These observations lead to the following definition.

Definition

A set A is **countable** if there exists a one–one map $f: A \to \mathbb{N}$, and is **uncountable** otherwise.

A countable set with infinitely many elements is **countably infinite**.

Remarks

(i) Since the definition does not require the one–one map to be *onto*, it follows that all finite sets (including $\varnothing$) are countable. For example, $\{7, \pi, \sqrt{2}, -31, \pi\sqrt{2}\}$ can be mapped one–one to $\{1, 2, 3, 4, 5\}$, a finite subset of $\mathbb{N}$, and so is countable.

(ii) If a set A is countable, so that there exists a one–one map from A to $\mathbb{N}$, then we can *list* its elements in the order defined by this one–one map. Conversely, if we can list the elements of a set A, then the ordering of the list defines a one–one map from A to $\mathbb{N}$. Hence a set is countable if and only if it is *listable*.

(iii) Every subset of a countable set is countable. However, a subset of an uncountable set may be countable or uncountable.

Some authors call a set A countable if there is a one–one *and onto* map from A to $\mathbb{N}$ — this alternative definition does *not* classify finite sets as countable.

We now investigate the countability of some of the standard subsets of $\mathbb{R}$. We begin by noting that the set $\mathbb{N}$ is clearly countable, since the map $f: \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f: n \longmapsto n$ is a one–one map from $\mathbb{N}$ to $\mathbb{N}$.

Problem 2.1

Show that the set $\mathbb{Z}$ of integers is countable.

You may find the next result surprising.

Theorem 2.1

The set $\mathbb{Q}$ of rational numbers is countable.

Proof There are several ways of proving this result. Recall that any non-zero rational number can be written uniquely as p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ have no common factors. If we agree to give 0 the representation $0/1$, we have a unique representation of the form p/q for each element of $\mathbb{Q}$. We now construct a one–one map from $\mathbb{Q}$ into $\mathbb{Z}$ as follows:

$$f: \frac{p}{q} \longmapsto \begin{cases} 2^p 3^q & \text{if } p \geq 0, \\ -2^{-p} 3^q & \text{if } p < 0. \end{cases}$$

For example, $f(3/4) = 2^3\,3^4 = 648$ and $f(-3/4) = -2^{-(-3)}\,3^4 = -648$.

We know that $\mathbb{Z}$ is countable (by Problem 2.1), and so there exists a one–one map from $\mathbb{Z}$ to $\mathbb{N}$. Composing these two maps, we obtain a one–one map from $\mathbb{Q}$ to $\mathbb{N}$. Thus $\mathbb{Q}$ is countable. ■

So far, we have looked only at countable subsets of $\mathbb{R}$, and you may be wondering whether *all* subsets of $\mathbb{R}$ are countable. In fact, there are infinitely many *un*countable subsets of $\mathbb{R}$. We now show that the subset of $\mathbb{R}$ comprising the open interval $(0, 1)$ is uncountable.

> ***Theorem 2.2***
>
> The set $(0,1)$ is uncountable.

Proof We prove this by supposing that $(0,1)$ is countable and obtaining a contradiction.

If $(0,1)$ is countable, then we can list all its elements in decimal form:

$$\begin{array}{lcl} x_1 & = & 0.\boxed{d_{1,1}}\, d_{1,2}\, d_{1,3} \dots \\ x_2 & = & 0.\, d_{2,1}\boxed{d_{2,2}}\, d_{2,3} \dots \\ x_3 & = & 0.\, d_{3,1}\, d_{3,2}\boxed{d_{3,3}} \dots \\ \vdots & \vdots & \vdots \end{array}$$

where each $d_{i,j}$ belongs to $\{0,1,\dots,9\}$.

Some numbers have two possible decimal expansions: for example, $0.1000\ldots = 0.0999\ldots$. In these cases, we here use only the expression ending in 0s.

We now choose a real number

$$x = 0.a_1a_2a_3\dots$$

in such a way that

$$a_1 \neq d_{1,1},\ a_2 \neq d_{2,2},\ a_3 \neq d_{3,3},\ \dots \text{ and } a_i \neq 9 \text{ for all } i.$$

We have now obtained a real number x in $(0,1)$ that does not appear in our list, since it differs from x_n at the nth decimal place. This, however, is a contradiction since the list was supposed to contain *all* the numbers in $(0,1)$. Thus our original supposition that $(0,1)$ is countable must be wrong, and so $(0,1)$ is uncountable. ■

By ensuring that $a_i \neq 9$ for all i, we have made certain that x does not end in an infinite sequence of 9s, and so is not equivalent to a number in our list ending in an infinite sequence of 0s.

> ***Corollary 2.3***
>
> The set $\mathbb{R}$ is uncountable.

Proof If $\mathbb{R}$ were countable, then the subset $(0,1)$ would be countable. This contradicts Theorem 2.2, and hence $\mathbb{R}$ is uncountable. ■

Problem 2.2

Show that any open interval (a,b) with $a<b$ is uncountable.

Hint Find a one–one map from $(0,1)$ to (a,b).

Remark

We can deduce from Problem 2.2 that open intervals of the form (a,∞) and $(-\infty,b)$ are uncountable, since each contains the uncountable subset (a,b).

When dealing with topological spaces, we frequently need to consider the union of two or more sets. In fact (perhaps surprisingly), the union of *countably* many countable sets is always countable.

> ***Theorem 2.4***
>
> For each $n \in \mathbb{N}$, let A_n be a countable set. Then $A = \bigcup_{n=1}^{\infty} A_n$ is also countable.

Proof Since each set A_n is countable, we can list the elements of A_1 as

$$x_{1,1}, x_{1,2}, x_{1,3}, \ldots,$$

the elements of A_2 as

$$x_{2,1}, x_{2,2}, x_{2,3}, \ldots,$$

and so on. We can then list all the elements of A as shown below.

$$\begin{array}{ccccccccc}
x_{1,1} & \rightarrow & x_{1,2} & & x_{1,3} & \rightarrow & x_{1,4} & & x_{1,5} \quad \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
x_{2,1} & & x_{2,2} & & x_{2,3} & & x_{2,4} & & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
x_{3,1} & & x_{3,2} & & x_{3,3} & & x_{3,4} & & \\
 & \swarrow & & \nearrow & & & & & \\
x_{4,1} & & x_{4,2} & & x_{4,3} & & x_{4,4} & & \cdots \\
\downarrow & \nearrow & & & & & & & \\
x_{5,1} & & \cdot & & \cdot & & \cdot & & \cdot \quad \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

We start our list with the element $x_{1,1}$, at the top left of the array (whose subscripts sum to 2). Next, we take the elements $x_{1,2}$ and $x_{2,1}$ on the diagonal immediately to the right of $x_{1,1}$ (whose subscripts sum to 3), then the elements on the next diagonal (whose subscripts sum to 4), and so on.

This gives us a list starting $x_{1,1}, x_{1,2}, x_{2,1}, x_{3,1}, x_{2,2}, x_{1,3}, x_{1,4}, \ldots$.

Every element $x_{i,j}$ appears somewhere in the list.

Thus A is listable, and therefore countable. ■

There is an obvious corollary, which is worth recording specifically.

> ***Corollary 2.5***
>
> The union of finitely many countable sets is countable.

2.2 Families of sets and index sets

We mentioned at the start of this section that, in topology, we have to deal with collections of sets. In mathematics, a collection of sets is often referred to as a **family of sets**. A family of sets is itself a set — a set of sets — and as such can be finite or infinite, countable or uncountable. Moreover, the sets in a family may be finite or infinite, countable or uncountable.

The term *family* is commonly used for collections of mathematical objects, and is not applied exclusively to collections of sets.

The simplest families of sets are *finite* families — for example, the indiscrete topology on any set X is $\{\varnothing, X\}$, a family of just two sets. In many cases, however, we need to consider *infinite* families of sets. For example, suppose that $X = \mathbb{R}$; then $\mathcal{A} = \{(-r,r) : r \in (0,\infty)\}$ and $\mathcal{B} = \{\{x\} : x \in \mathbb{Q}\}$ are infinite families of subsets of X. The family $\mathcal{A}$ is uncountable whereas $\mathcal{B}$ is countably infinite. Moreover, the sets in $\mathcal{A}$ are infinite and uncountable whereas those in $\mathcal{B}$ are finite.

Families have *subfamilies* in the same way that sets have subsets. For example, $\mathcal{C} = \{(-n,n) : n \in \mathbb{N}\}$ is a subfamily of $\mathcal{A} = \{(-r,r) : r \in (0,\infty)\}$.

The elements of a family of sets are often conveniently *indexed* by the elements of another set. For example, the families $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ above are indexed by the sets $(0, \infty)$, $\mathbb{Q}$ and $\mathbb{N}$ respectively. Similarly, the sets A_n in Theorem 2.4 form a family of sets indexed by $\mathbb{N}$.

Definition

A set I is an **index set** of a family $\mathcal{F}$ of sets if each set in $\mathcal{F}$ can be labelled as A_i, for some $i \in I$, and if, for each $i \in I$, there is a set in $\mathcal{F}$ indexed by i. Thus,

$$\mathcal{F} = \{A_i : i \in I\}.$$

Remarks

(i) We do not insist on a one–one correspondence between an index set and the family of sets that it indexes. For example, for each $n \in \mathbb{N}$ let P_n be the set of prime numbers that divide n. Then $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ is an indexed family of sets; but the correspondence between the index set $\mathbb{N}$ and the family $\mathcal{P}$ is not one–one: for example, $P_6 = P_{12} = \{2, 3\}$.

(ii) If an index set I is finite then so is any family $\mathcal{F}$ indexed by I, and if I is countable then so is $\mathcal{F}$. However, if I is countably infinite then $\mathcal{F}$ could be finite or countably infinite, and if I is uncountable then $\mathcal{F}$ could be finite, countably infinite or uncountable.

In many of the applications in this course, however, there is a one–one correspondence between an index set I and a family $\mathcal{F}$ indexed by I, in which case we *can* say that if I is countably infinite then so is $\mathcal{F}$, and if I is uncountable then so is $\mathcal{F}$.

(iii) Unsurprisingly, the set $\mathbb{N}$ is commonly used as an index set for any countably infinite family of sets.

An example of a family of sets indexed by a finite set is the family $\mathcal{F}$ of all subsets of $\{a, b, c\}$, whose eight members can be indexed by the set $I = \{1, 2, \ldots, 8\}$ as follows:

$$A_1 = \varnothing, \quad A_2 = \{a\}, \quad A_3 = \{b\}, \quad A_4 = \{c\},$$
$$A_5 = \{a, b\}, \quad A_6 = \{b, c\}, \quad A_7 = \{a, c\}, \quad A_8 = \{a, b, c\}.$$

Examples of families of sets indexed by infinite sets are the families $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ discussed earlier and the families $\mathcal{G} = \{\{-n, n\} : n \in \mathbb{N}\}$ and $\mathcal{H} = \{(r, r+1) : r \in \mathbb{R}\}$. In the case of $\mathcal{G}$, $\mathbb{N}$ is countably infinite and the indexing is one–one, so $\mathcal{G}$ is a countably infinite family of sets. Similarly, $\mathcal{B}$ and $\mathcal{C}$ are countably infinite families of sets. In the case of $\mathcal{H}$, $\mathbb{R}$ is uncountable and the indexing is one–one, so $\mathcal{H}$ is an uncountable family of sets. Similarly, $\mathcal{A}$ is an uncountable family of sets.

$\mathcal{A} = \{(-r, r) : r \in (0, \infty)\}$.
$\mathcal{B} = \{\{x\} : x \in \mathbb{Q}\}$.
$\mathcal{C} = \{(-n, n) : n \in \mathbb{N}\}$.

Problem 2.3

For each of the following families of sets, write down the index set for the family and state whether the family is finite, countably infinite or uncountable.

(a) $\{[x, x + \pi] : x \in \mathbb{Q}\}$

(b) $\{(i - 1, i + 1) : i \in \{1, 2, 3, 4\}\}$

(c) $\{\{x\} : x \in \mathbb{R}\}$

Problem 2.4

Let $\mathcal{F}$ be the family of all closed intervals in $\mathbb{R}$ of unit length and with rational endpoints. Write $\mathcal{F}$ as an indexed family, and state whether $\mathcal{F}$ is finite, countably infinite or uncountable.

The index notation gives us a neat way of writing unions and intersections.

> **Definition**
>
> Let $\mathcal{F} = \{A_i : i \in I\}$ be a family of subsets of a given set X.
>
> The **union** of the sets in $\mathcal{F}$, written $\bigcup_{i\in I} A_i$, is defined by
>
> $$\bigcup_{i\in I} A_i = \{x \in X : x \in A_i \text{ for } \textit{some } i \in I\}.$$
>
> The **intersection** of the sets in $\mathcal{F}$, written $\bigcap_{i\in I} A_i$, is defined by
>
> $$\bigcap_{i\in I} A_i = \{x \in X : x \in A_i \text{ for } \textit{all } i \in I\}.$$

You may write the subscript '$i \in I$' underneath or beside the union or intersection symbol. We use both.

Worked problem 2.1

Let $A_i = (i-1, i+1)$, for each $i \in \mathbb{N}$. Determine $\bigcup_{i\in\mathbb{N}} A_i$ and $\bigcap_{i\in\mathbb{N}} A_i$.

Solution

We have

$$A_1 = (0,2),\ A_2 = (1,3),\ A_3 = (2,4),\ \ldots\,.$$

Consider the union first. In each set A_i, all the elements are positive numbers, and so the union of these sets is contained in $(0,\infty)$, i.e. $\bigcup_{i\in\mathbb{N}} A_i \subseteq (0,\infty)$.

We now need to decide whether $\bigcup_{i\in\mathbb{N}} A_i$ is *equal* to $(0,\infty)$. By looking at the sets A_1, A_2 and A_3, we see that each set in the family overlaps with other sets below and above it; this suggests that $\bigcup_{i\in\mathbb{N}} A_i = (0,\infty)$.

Suppose that $x \in (0,\infty)$ and x is *not* an integer. Then there is an integer i such that $i < x < i+1$, and so $x \in (i, i+1) \subset A_{i+1}$, as Figure 2.1 illustrates.

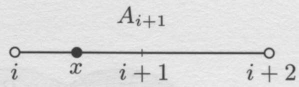

Figure 2.1

Now suppose that $x \in (0,\infty)$ and x *is* an integer. Then, $x \in (x-1, x+1) = A_x$.

Thus, every element of $(0,\infty)$ belongs to the union, i.e. $(0,\infty) \subseteq \bigcup_{i\in\mathbb{N}} A_i$.

Therefore $\bigcup_{i\in\mathbb{N}} A_i = (0,\infty)$.

Now consider the intersection. A point x belongs to $\bigcap_{i\in\mathbb{N}} A_i$ if it belongs to *all* of the sets A_i. But there are no such points since, if $x \in A_i$ for some $i \in \mathbb{N}$, then $x < i+1$ and so x does not belong to $A_{i+2} = (i+1, i+3)$; for example, $A_1 \cap A_3 = \varnothing$.

Thus, $\bigcap_{i\in\mathbb{N}} A_i = \varnothing$. ■

Problem 2.5

Determine $\bigcup_{i\in I} A_i$ and $\bigcap_{i\in I} A_i$, in each of the following cases:
(a) $I = \mathbb{Q}$ and $A_i = \{i\}$ for each $i \in I$;
(b) $I = (0, \infty)$ and $A_i = (-i, i)$ for each $i \in I$.

2.3 Unions and intersections

This section concludes with some results on unions and intersections of sets that will be useful in our study of topological spaces. We omit the proofs.

We begin with the *Distributive Laws.* In the simplest case, when we have just three sets A, B and C, the Distributive Laws have the following form:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C);$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

These are illustrated in Figures 2.2 and 2.3 respectively.

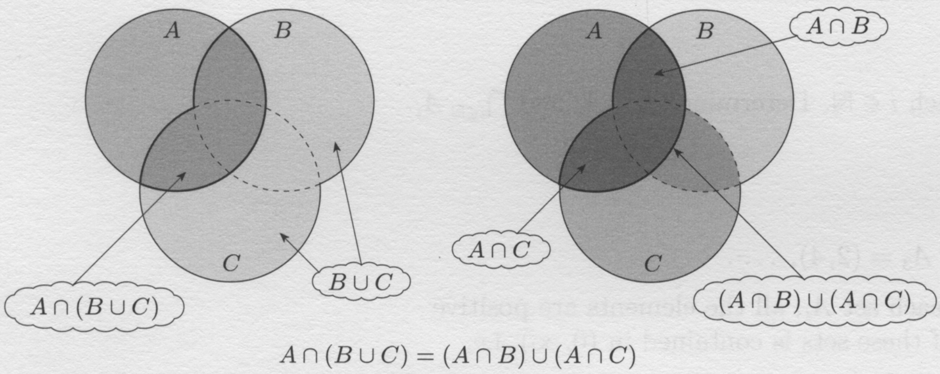

Figure 2.2

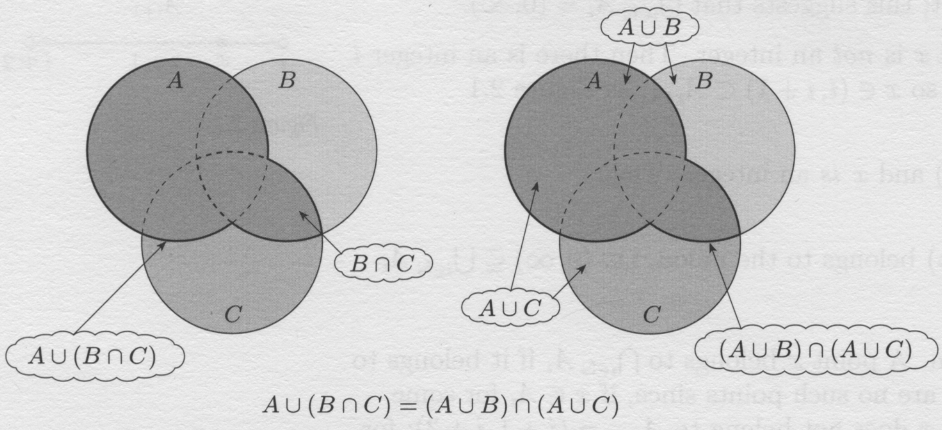

Figure 2.3

More generally, the following are true.

Theorem 2.6 Distributive Laws

Let $\{A_i : i \in I\}$ and $\{B_j : j \in J\}$ be families of sets. Then

$$\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{j\in J} B_j\right) = \bigcup_{i\in I, j\in J} (A_i \cap B_j),$$

$$\left(\bigcap_{i\in I} A_i\right) \cup \left(\bigcap_{j\in J} B_j\right) = \bigcap_{i\in I, j\in J} (A_i \cup B_j).$$

Since we are concerned with continuity in topological spaces, we shall often be interested in images and inverse images of sets that are defined as unions and intersections of families of sets.

Let f be any function from a set X to a set Y, and let A_1 and A_2 be subsets of X. Consider the image sets $f(A_1)$ and $f(A_2)$. You may think it fairly evident that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$, but what about $f(A_1 \cap A_2)$?

Problem 2.6

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, and let $A_1 = [-1, 0]$ and $A_2 = [0, 2]$. Show that $f(A_1 \cap A_2)$ is a proper subset of $f(A_1) \cap f(A_2)$.

Recall that a subset A of a set X is *proper* if $A \neq X$.

Problem 2.6 demonstrates that it is not generally true that $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. However, if $x \in A_1 \cap A_2$, then $x \in A_1$ so that $f(x) \in f(A_1)$ and $x \in A_2$ so that $f(x) \in f(A_2)$. Thus, $f(x) \in f(A_1) \cap f(A_2)$, and so we *can* assert that

$$f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2).$$

For inverse images, however, we do not have to be so cautious. Let B_1 and B_2 be subsets of Y. You may think it fairly evident that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. Moreover, the inverse image of $B_1 \cap B_2$ is the set of all $x \in X$ such that $f(x)$ belongs to $B_1 \cap B_2$, and therefore *both* to B_1 *and* to B_2. Thus

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2).$$

We can generalize these results to unions and intersections of families of any number of sets.

Theorem 2.7 Functions of unions

Let $f\colon X \to Y$, let $\{A_i : i \in I\}$ be a family of subsets of X, and let $\{B_j : j \in J\}$ be a family of subsets of Y. Then:

Forward mapping

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i);$$

Inverse mapping

$$f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}(B_j).$$

> ***Theorem 2.8 Functions of intersections***
>
> Let $f: X \to Y$, let $\{A_i : i \in I\}$ be a family of subsets of X, and let $\{B_j : j \in J\}$ be a family of subsets of Y. Then:
>
> *Forward mapping*
>
> $$f\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} f(A_i);$$
>
> *Inverse mapping*
>
> $$f^{-1}\left(\bigcap_{j\in J} B_j\right) = \bigcap_{j\in J} f^{-1}(B_j).$$

Note that we have equality in all cases except for the forward mapping of intersections.

Next, there are Distributive Laws for intersection over products and union over products.

> ***Theorem 2.9 Intersection is distributive over products***
>
> Let X and Y be sets, let $A_1, A_2 \subseteq X$ and let $B_1, B_2 \subseteq Y$. Then
>
> $$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2).$$

Finally, we shall need *laws* concerning the *complements* of sets.

> ***Definition***
>
> Let A be a subset of X. The **complement** of A with respect to X is $A^c = X - A$, the set of elements in X but not in A.

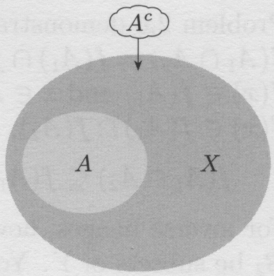

Figure 2.4

Remarks

(i) The meaning of A^c depends on the context: we need to know the set X with respect to which we are taking the complement.

(ii) We can deduce from the definition that $(A^c)^c = A$.

Problem 2.7

Let $X = \mathbb{R}$ and $A = [0, 1]$. Write the complement A^c of A with respect to X as a union of intervals.

Problem 2.8

Write down the complement A^c of $A = \{1, 3\}$ with respect to X when:

(a) $X = \{1, 2, 3\}$; (b) $X = \{1, 2, 3, 4, 5\}$.

The laws we require are known as *De Morgan's Laws.* In the simplest case, when we have just two sets A_1 and A_2, they take the following form:

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c;$$
$$(A_1 \cap A_2)^c = A_1^c \cup A_2^c.$$

Augustus De Morgan (1806–71) contributed to set theory and mathematical logic, and was a distinguished popularizer of mathematics.

These are illustrated in Figures 2.5 and 2.6.

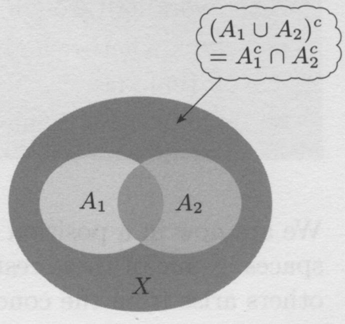

Figure 2.5

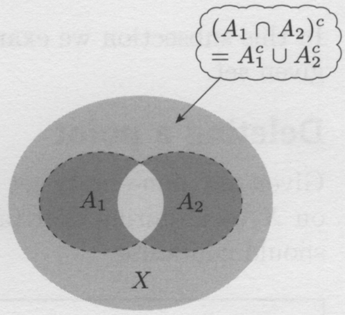

Figure 2.6

Problem 2.9

For $X = \{1, 2, \ldots, 12\}$, let $A_1 = \{x \in X : x \text{ is even}\}$ and let $A_2 = \{x \in X : x \text{ is divisible by } 3\}$. Verify De Morgan's Laws for the sets A_1 and A_2.

More generally, De Morgan's Laws have the following form.

> ***Theorem 2.10 De Morgan's Laws***
>
> Let $\{A_i : i \in I\}$ be a family of subsets of X. Then:
>
> *First law*
>
> $$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c;$$
>
> *Second law*
>
> $$\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c.$$

Problem 2.10

Write out De Morgan's Laws when the index set is $I = \{1, 2, 3\}$.

---

3 Examples of topological spaces

After working through this section, you should be able to:

- check whether a given family of subsets of a set defines a topology on that set;
- explain what is meant by a *subspace* of a topological space.

We are now in a position to produce further examples of topological spaces. Some of these result from rather surprising constructions, whereas others arise from the concept of a *subspace* of a topological space.

3.1 Constructions for topological spaces

In this subsection we examine four ways of constructing a topology on a given set.

Deleting a point

Given any non-empty set X and any point $a \in X$, we can define a topology on X by requiring that (apart from X itself) *no* open set of the topology should contain a.

Definition

Let X be a non-empty set and let $a \in X$. The **a-deleted-point topology** on X is

$$\mathcal{T}_a = \{X\} \cup \{U \subseteq X : a \notin U\}.$$

Remark

We do not delete a from the set on which the topology is defined. The point a is still in the original set X.

To verify that $\mathcal{T}_a$ is indeed a topology on X, we must show that $\mathcal{T}_a$ satisfies conditions (T1)–(T3). Before doing this, we ask you to investigate a particular deleted-point topology.

Problem 3.1

Let $\mathcal{T}_1$ be the 1-deleted-point topology on $\mathbb{R}$. Which of the following sets belong to $\mathcal{T}_1$?

$$[0,2], \quad \mathbb{R}, \quad (0,1), \quad \mathbb{N}, \quad (-\infty,0], \quad \{2,3,4\}.$$

We now verify that $\mathcal{T}_a$ is a topology on X.

Worked problem 3.1

Let X be a non-empty set and let $a \in X$. Show that

$$\mathcal{T}_a = \{X\} \cup \{U \subseteq X : a \notin U\}$$

is a topology on X.

Solution

We must show that $\mathcal{T}_a$ satisfies (T1)–(T3).

(T1) By definition, $X \in \mathcal{T}_a$. Also, $a \notin \varnothing$ and so $\varnothing \in \mathcal{T}_a$. Thus (T1) is satisfied.

(T2) Let $U_1, U_2 \in \mathcal{T}_a$ and let $U = U_1 \cap U_2$. We must show that $U \in \mathcal{T}_a$.

If U_1 is equal to X then $U = X \cap U_2 = U_2 \in \mathcal{T}_a$.

The other possibility is that $U_1 \neq X$; then, by the definition of $\mathcal{T}_a$, we have $a \notin U_1$, and so $a \notin U_1 \cap U_2 = U$. Therefore $U \in \mathcal{T}_a$.

Thus (T2) is satisfied.

(T3) Let $\{U_i : i \in I\}$ be a family of sets in $\mathcal{T}_a$ and let $U = \bigcup_{i \in I} U_i$. We must show that $U \in \mathcal{T}_a$.

If $U_j = X$, for some $j \in I$, then $U = X \in \mathcal{T}_a$.

The other possibility is that none of the U_i is equal to X. By the definition of $\mathcal{T}_a$, we then have $a \notin U_i$, for each $i \in I$, and so $a \notin \bigcup_{i \in I} U_i = U$. Therefore $U \in \mathcal{T}_a$.

Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}_a$ is a topology on X. ■

The co-finite topology

Let X be an infinite set. It may seem reasonable to define X and all *finite* subsets of X as the open sets of a topology. However, this is not possible, because (T3) requires the union of *infinitely* many open sets to be open, and such a union is generally an infinite set.

Suppose, though, that we take all the *complements* of finite sets in X. Does this family define a topology? The answer is yes.

> ***Definition***
>
> Let X be a set. The **co-finite topology** on X is
>
> $$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X : U^c \text{ is finite}\}.$$

Before showing you that this does indeed define a topology, we ask you to investigate some of the open sets for the co-finite topology on $\mathbb{N}$.

Problem 3.2

Which of the following sets belong to the co-finite topology on $\mathbb{N}$?

$$\{1,2\}, \quad \{k \in \mathbb{N} : k \geq 3\}, \quad \{2k : k \in \mathbb{N}\}, \quad \varnothing.$$

Worked problem 3.2

Let X be any set. Show that

$$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X : U^c \text{ is finite}\}$$

is a topology on X.

Solution

We must show that $\mathcal{T}$ satisfies (T1)–(T3).

(T1) By definition, $\varnothing \in \mathcal{T}$. Also, $X^c = \varnothing$ is finite, and so $X \in \mathcal{T}$. Thus (T1) is satisfied.

(T2) Let $U_1, U_2 \in \mathcal{T}$ and let $U = U_1 \cap U_2$. We must show that $U \in \mathcal{T}$.

If U_1 or U_2 is equal to $\varnothing$, then $U = \varnothing$, which is in $\mathcal{T}$.

Otherwise, both U_1^c and U_2^c are finite. By De Morgan's Second Law, $U^c = U_1^c \cup U_2^c$ is the union of two finite sets, and is therefore finite. Hence $U \in \mathcal{T}$.

Thus (T2) is satisfied.

(T3) Let $\{U_i : i \in I\}$ be a family of sets in $\mathcal{T}$ and let $U = \bigcup_{i\in I} U_i$. We must show that $U \in \mathcal{T}$.

If $U_i = \varnothing$, for each $i \in I$, then $U = \varnothing$, which is in $\mathcal{T}$.

The other possibility is that U_j^c is finite, for some $j \in I$. By De Morgan's First Law, $U^c = \bigcap_{i\in I} U_i^c$. Thus $U^c \subseteq U_j^c$, and so U^c is finite, proving that $U \in \mathcal{T}_a$.

Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}$ is a topology on X. ■

The co-countable topology

The next example is similar to the co-finite topology, but this time the open sets are the sets with *countable* complements.

> ### *Definition*
>
> Let X be a set. The **co-countable topology** on X is
>
> $$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X : U^c \text{ is countable}\}.$$

For example, the co-countable topology on $\mathbb{Z}$ consists of the empty set together with all subsets of $\mathbb{Z}$ with countable complements. Since $\mathbb{Z}$ is itself countable, this is in fact *all* the subsets of $\mathbb{Z}$, and so is the discrete topology on $\mathbb{Z}$. However, the co-countable topology on $\mathbb{R}$ is more interesting.

Problem 3.3

Which of the following sets belong to the co-countable topology on $\mathbb{R}$?

$$\mathbb{R} - \mathbb{Z},\quad (-\infty, 0],\quad (-\infty, 0) \cup (0, \infty),\quad \{2, 3, 4\}.$$

Problem 3.4

Let X be any set. Show that

$$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X : U^c \text{ is countable}\}$$

is a topology on X.

Either-or topologies

An **either-or topology** $\mathcal{T}$ on a set X is one that can be defined as the union of two distinct families $\mathcal{F}_1$ and $\mathcal{F}_2$ of subsets of X. Thus a set U belongs to $\mathcal{T}$ if and only if it belongs to $\mathcal{F}_1$ or to $\mathcal{F}_2$.

Problem 3.5

Let $X = [-1, 1]$ and let $\mathcal{T} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{U \subseteq X : 0 \notin U\} \quad \text{and} \quad \mathcal{F}_2 = \{U \subseteq X : (-1, 1) \subseteq U\}.$$

Which of the following sets belong to $\mathcal{T}$?

$$[0, 1], \ [-1, 1], \ \{x : x \in \mathbb{Q}, |x| < 1\}, \ (-1, 0), \ (-1, 1].$$

Worked problem 3.3

Let $X = [-1, 1]$ and let $\mathcal{T}$ be the family of subsets of X defined in Problem 3.5. Show that $\mathcal{T}$ is a topology on X.

Solution

We must show that $\mathcal{T}$ satisfies (T1)–(T3).

(T1) The set $\varnothing$ belongs to $\mathcal{F}_1$ and the set X belongs to $\mathcal{F}_2$, so (T1) is satisfied.

(T2) Let $U_1, U_2 \in \mathcal{T}$ and let $U = U_1 \cap U_2$. We must show that $U \in \mathcal{T}$.

First, suppose that at least one of the sets U_1 and U_2 (say U_1) belongs to $\mathcal{F}_1$. Then $0 \notin U_1$, and so $0 \notin U$, so that $U \in \mathcal{F}_1 \subseteq \mathcal{T}$.

The other possibility is that both U_1 and U_2 belong to $\mathcal{F}_2$. Then $(-1, 1) \subseteq U_1$ and $(-1, 1) \subseteq U_2$. It follows that $(-1, 1) \subseteq U$, so that $U \in \mathcal{F}_2 \subseteq \mathcal{T}$.

Thus (T2) is satisfied.

(T3) Let $\{U_i : i \in I\}$ be a family of sets in $\mathcal{T}$ and let $U = \bigcup_{i \in I} U_i$. We must show that $U \in \mathcal{T}$.

First we suppose that, for each $i \in I$, $U_i \in \mathcal{F}_1$ and so $0 \notin U_i$. It follows that $0 \notin U$, so that $U \in \mathcal{F}_1 \subseteq \mathcal{T}$.

The other possibility is that $U_j \in \mathcal{F}_2$ for some $j \in I$, so that $(-1, 1) \subseteq U_j$. It follows that $(-1, 1) \subseteq U$. Hence $U \in \mathcal{F}_2 \subseteq \mathcal{T}$.

Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}$ is a topology on X. ■

Problem 3.6

Let $X = \mathbb{R}$ and let $\mathcal{T} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{U \subseteq X : 0 \notin U\} \quad \text{and} \quad \mathcal{F}_2 = \{U \subseteq X : 0 \in U \text{ and } U^c \text{ is finite}\}.$$

(a) Which of the following sets belong to $\mathcal{T}$?

$$\mathbb{Q}, \ (-\infty, 1) \cup (1, \infty), \ \{x \in \mathbb{N} : x \geq 10\}, \ \mathbb{R} - \mathbb{Z}.$$

(b) Show that $\mathcal{T}$ is a topology on X.

Non-metrizable topologies

We conclude this subsection by showing that at least two of the new topologies we have just seen are not metrizable — that is, they do not comprise the d-open sets for some metric d. Thus we have added to our collection of topological spaces that are not metric spaces.

The concept of metrizability was introduced in Section 1.

Worked problem 3.4

Let X be a set containing at least two elements and let $a \in X$. Show that the a-deleted-point topology $\mathcal{T}_a$ on X is not metrizable.

Recall that $U \in \mathcal{T}_a$ if $a \notin U$ or if $U = X$.

Solution

We suppose that $\mathcal{T}_a$ is metrizable and obtain a contradiction. That is, we suppose that there exists a metric d on X such that $\mathcal{T}_a = \mathcal{T}(d)$ — in other words, such that the sets in $\mathcal{T}_a$ are the d-open subsets of X. Thus, for any $r > 0$, the d-open ball $B_d(a, r)$ belongs to $\mathcal{T}_a$.

Now let $b \in X$ with $b \neq a$, and let $r = \frac{1}{2}d(a,b)$. Then $a \in B_d(a,r)$ but $b \notin B_d(a,r)$, and so $B_d(a,r)$ is a d-open set containing a but not equal to X. Thus, $B_d(a,r)$ belongs to $\mathcal{T}(d)$ but not to $\mathcal{T}_a$, contradicting the assumption that $\mathcal{T}_a = \mathcal{T}(d)$.

Thus $\mathcal{T}_a$ is not metrizable. ■

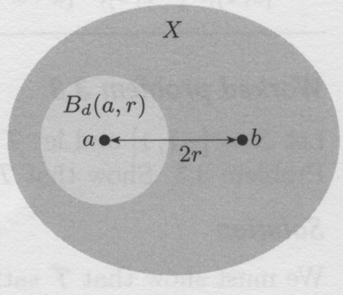

Figure 3.1

Problem 3.7

Let X be an infinite set and let $\mathcal{T}$ be the co-finite topology on X.

Recall that $U \in \mathcal{T}$ if U^c is finite or if $U = \varnothing$.

(a) Let U_1 and U_2 be non-empty $\mathcal{T}$-open subsets of X. Show that $U_1 \cap U_2 \neq \varnothing$.

(b) By considering two open balls of small radii centred at distinct points of X, show that $\mathcal{T}$ is not metrizable.

3.2 Subspaces

You have seen that, for a metric space (X,d), any subset $A \subseteq X$ can be given a metric of its own, inherited from d; the resulting metric space (A, d_A) is known as a *metric subspace* of (X,d).

Unit A2, Subsection 3.1.

There is a corresponding notion for a topological space $(X, \mathcal{T})$. Let A be a subset of X. There is no metric to restrict from X to A, but there are open sets that can be restricted: if U is an open subset of X, then $U \cap A$ is a subset of A. So let us consider the sets $U \cap A$, for all $U \in \mathcal{T}$.

For example, let $X = \{a,b,c,d\}$ and $A = \{a,b\}$, and consider the topology

$$\mathcal{T} = \{\varnothing,\ \{c\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\},\ X\}$$

You may like to check that $\mathcal{T}$ is a topology on X.

on X. Taking the intersection of each of the sets in $\mathcal{T}$ with the set A, we obtain

$$\{\varnothing,\ \varnothing,\ \{a\},\ \{b\},\ \{a,b\},\ A\} = \{\varnothing,\ \{a\},\ \{b\},\ A\},$$

which is a topology on A (in fact, the discrete topology).

We now prove that the family of sets obtained in this way always gives a topology on $A \subseteq X$.

Theorem 3.1

Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. Then the family

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$$

is a topology on A.

Proof We must show that $\mathcal{T}_A$ satisfies (T1)–(T3).

(T1) Since $\varnothing \in \mathcal{T}$ and $\varnothing \cap A = \varnothing$, we have $\varnothing \in \mathcal{T}_A$. Since $X \in \mathcal{T}$ and $X \cap A = A$, we have $A \in \mathcal{T}_A$. Thus (T1) is satisfied.

Thus A is $\mathcal{T}_A$-open even if it is not $\mathcal{T}$-open.

(T2) Let $V_1, V_2 \in \mathcal{T}_A$ and let $V = V_1 \cap V_2$. We must show that $V \in \mathcal{T}_A$.

Since $V_1, V_2 \in \mathcal{T}_A$, there are sets $U_1, U_2 \in \mathcal{T}$ such that $V_1 = U_1 \cap A$ and $V_2 = U_2 \cap A$. Then

$$V = (U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A.$$

Since $U_1, U_2 \in \mathcal{T}$, we have $U_1 \cap U_2 \in \mathcal{T}$ and hence $V \in \mathcal{T}_A$.

Thus (T2) is satisfied.

(T3) Let $\{V_i : i \in I\}$ be a family of sets in $\mathcal{T}_A$ and let $V = \bigcup_{i \in I} V_i$. We must show that $V \in \mathcal{T}_A$.

From the definition of $\mathcal{T}_A$, for each $i \in I$ there exists $U_i \in \mathcal{T}$ such that $V_i = U_i \cap A$. We now use the first of the Distributive Laws for families, in the special case where the second family consists of a single set:

Theorem 2.6.

$$V = \bigcup_{i \in I} V_i = \bigcup_{i \in I} (U_i \cap A) = \left(\bigcup_{i \in I} U_i \right) \cap A.$$

Since $\bigcup_{i \in I} U_i$ belongs to $\mathcal{T}$ (by property (T3) for $\mathcal{T}$), it follows that $V \in \mathcal{T}_A$.

Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}_A$ is a topology on A. ∎

Definition

Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. Then the family

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$$

is the **subspace topology** on A inherited from $\mathcal{T}$, and the topological space $(A, \mathcal{T}_A)$ is a **(topological) subspace** of $(X, \mathcal{T})$.

Problem 3.8

Let $X = \{a, b, c, d, e\}$ and consider the topology on X given by

$$\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, e\}, \{a, b, c, d\}, X\}.$$

Find the subspace topology on $A = \{a, b, c\}$ inherited from $\mathcal{T}$.

You may assume that $\mathcal{T}$ is a topology on X.

Remark

For each of the examples in this subsection, there are several sets (including the set A) that are open for the subspace topology but not for the original topology. It is important to remember that this can happen when working with subspaces.

We now look at some subspace topologies inherited from the deleted-point topologies.

Worked problem 3.5

Let $\mathcal{T}_0$ be the 0-deleted-point topology on $\mathbb{R}$. Determine the subspace topology on $\mathbb{Z}$ inherited from $\mathcal{T}_0$.

Solution

The sets in $\mathcal{T}_0$ are all the subsets of $\mathbb{R}$ that do not contain 0, together with $\mathbb{R}$. Taking the intersections of these sets with $\mathbb{Z}$ gives all the subsets of $\mathbb{Z}$ that do not contain 0, together with $\mathbb{Z}$. So the subspace topology on $\mathbb{Z}$ is the 0-deleted-point topology on $\mathbb{Z}$. ■

Problem 3.9

Let $\mathcal{T}_a$ be the a-deleted-point topology on an infinite set X and let A be a subset of X that does not contain a. Determine the subspace topology on A inherited from $\mathcal{T}_a$.

4 Continuity for topological spaces

After working through this section, you should be able to:

- use the definition of *continuity* for topological spaces;
- explain what is meant by a *homeomorphism*.

Our original aim in introducing topological spaces was to obtain a general class of spaces for which a notion of continuity could be defined. In Section 1 we established the following definition of continuity for functions between topological spaces.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A function $f\colon X \to Y$ is **continuous** if $f^{-1}(U) \in \mathcal{T}_X$ whenever $U \in \mathcal{T}_Y$.

We also looked at some examples of continuous functions between sets with finitely many elements.

We begin this section with some further examples of continuous functions. We then study the particular case of an invertible function f for which both f and its inverse are continuous, and establish some results that will be useful when we study the topology of surfaces in Block B.

4.1 Examples of continuous functions

In *Unit A2* you saw that the continuity of a function between metric spaces depends on which metrics are used for the domain and the codomain. Similarly, the continuity of a function between topological spaces depends on the topologies on the domain and the codomain. We begin this subsection with some simple examples of functions between sets, and investigate which topologies on the domain and the codomain make these functions continuous between the corresponding topological spaces.

Our first example is the *constant function* which maps each element of the domain to the same point in the codomain.

Definition

Let X and Y be sets. Then $f\colon X \to Y$ is a **constant function** if there exists $c \in Y$ such that $f(x) = c$ for each $x \in X$.

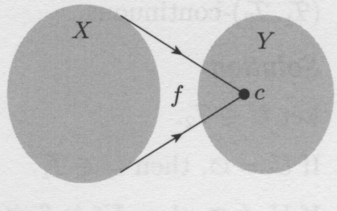

Figure 4.1

We now show that a constant function from X to Y is continuous *whatever* topologies are on X and Y.

Theorem 4.1

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $f: X \to Y$ be a constant function. Then f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous.

Proof Since f is a constant function, there exists $c \in Y$ such that $f(x) = c$ for each $x \in X$. Now let $U \in \mathcal{T}_Y$. We must show that $f^{-1}(U) \in \mathcal{T}_X$. We consider two cases.

If $c \notin U$, then $f^{-1}(U) = \varnothing$, since there are no points of X that map to U.

If $c \in U$, then $f^{-1}(U) = X$, since every point of X maps to c.

Since both $\varnothing$ and X belong to $\mathcal{T}_X$, it follows from the definition of continuity that f is continuous. ■

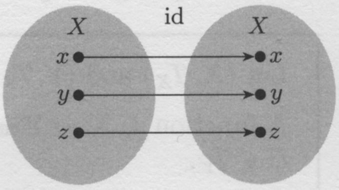

Figure 4.2

The symbol id is used for the identity function on X, whatever set X it refers to. If we need to avoid ambiguity, we write id_X.

We now consider another simple function, the *identity function*, which fixes each element of the domain.

Definition

Let X be a set. The **identity function** on X is the map $\mathrm{id}: X \to X$ defined by $\mathrm{id}(x) = x$, for all $x \in X$.

The identity function is continuous only if the codomain topology is a subset of the domain topology, as we now show.

Theorem 4.2

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X. The identity function on X is $(\mathcal{T}_1, \mathcal{T}_2)$-continuous if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Proof We observe first that, for each subset U of X, $\mathrm{id}^{-1}(U) = U$. It follows from the definition of continuity that id is $(\mathcal{T}_1, \mathcal{T}_2)$-continuous if and only if $\mathrm{id}^{-1}(U) \in \mathcal{T}_1$ whenever $U \in \mathcal{T}_2$ — that is, if and only if $U \in \mathcal{T}_1$ whenever $U \in \mathcal{T}_2$. Thus, id is $(\mathcal{T}_1, \mathcal{T}_2)$-continuous if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$. ■

Worked problem 4.1

Let X be a set, let $\mathcal{T}_1$ be the co-countable topology on X, and let $\mathcal{T}_2$ be the co-finite topology on X. Determine whether the identity function on X is $(\mathcal{T}_1, \mathcal{T}_2)$-continuous.

Recall that $U \in \mathcal{T}_1$ if U^c is countable or $U = \varnothing$, and that $U \in \mathcal{T}_2$ if U^c is finite or $U = \varnothing$.

Solution

Let $U \in \mathcal{T}_2$.

If $U = \varnothing$, then $U \in \mathcal{T}_1$.

If $U \neq \varnothing$, then U^c is finite, and hence countable. Thus $U \in \mathcal{T}_1$.

It follows that $\mathcal{T}_2 \subseteq \mathcal{T}_1$, and so (by Theorem 4.2) the identity function on X is $(\mathcal{T}_1, \mathcal{T}_2)$-continuous. ■

Problem 4.1

Let $X = \{a, b, c\}$ and let

$$\mathcal{T}_1 = \{\varnothing, \{a\}, X\}, \quad \mathcal{T}_2 = \{\varnothing, X\} \quad \text{and} \quad \mathcal{T}_3 = \{\varnothing, \{a\}, \{a, b\}, X\}$$

be topologies on X. For which values of i and j in $\{1, 2, 3\}$ is the identity function on X $(\mathcal{T}_i, \mathcal{T}_j)$-continuous?

You may assume that $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are topologies on X.

Our next function is the *characteristic function* of a set.

Definition

Let X be a set and let S be a subset of X. The **characteristic function** of S is the function $\chi_S : X \to \{0, 1\}$ defined by

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \in S^c. \end{cases}$$

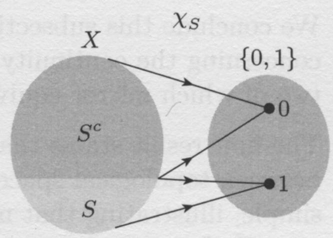

Figure 4.3

We now show that, if we have the discrete topology on $\{0, 1\}$, then the characteristic function of $S \subseteq X$ is continuous precisely when S and S^c both belong to the topology on X.

Theorem 4.3

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, where $Y = \{0, 1\}$ and $\mathcal{T}_Y$ is the discrete topology on Y, and let S be a subset of X. Then the characteristic function $\chi_S : X \to Y$ is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous if and only if both S and S^c belong to $\mathcal{T}_X$.

Proof The discrete topology on $Y = \{0, 1\}$ is the collection of all the subsets of $\{0, 1\}$: that is, $\mathcal{T}_Y = \{\varnothing, \{0\}, \{1\}, Y\}$. Now let $U \in \mathcal{T}_Y$. We consider each of the four possibilities for $\chi_S^{-1}(U)$. We obtain:

$$\chi_S^{-1}(U) = \begin{cases} S & \text{if } U = \{1\}, \\ S^c & \text{if } U = \{0\}, \\ X & \text{if } U = Y, \\ \varnothing & \text{if } U = \varnothing. \end{cases}$$

It follows from the definition of continuity that χ_S is continuous if and only if $\chi_S^{-1}(U) \in \mathcal{T}_X$ in each of these cases. We know that by definition $\varnothing$ and X belong to $\mathcal{T}_X$, and so f is continuous if and only if both S and S^c belong to $\mathcal{T}_X$. ■

In the following problems, we ask you to show that changing the topology on Y changes the conditions under which the characteristic function is continuous.

Problem 4.2

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, where $Y = \{0,1\}$ and $\mathcal{T}_Y = \{\varnothing, \{1\}, Y\}$. Let S be any subset of X. Show that the characteristic function $\chi_S : X \to Y$ is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous if and only if $S \in \mathcal{T}_X$.

You saw earlier that $\mathcal{T}_Y$ is a topology on Y.

Problem 4.3

Let $Y = \{0,1\}$. Give an example of a topological space $(X, \mathcal{T}_X)$, a subset S of X, and two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on Y, such that the characteristic function $\chi_S : X \to Y$ is $(\mathcal{T}_X, \mathcal{T}_1)$-continuous but not $(\mathcal{T}_X, \mathcal{T}_2)$-continuous.

We conclude this subsection with three straightforward and useful results concerning the continuity of functions between topological spaces, the first two of which mirror equivalent results for real functions.

The first result states that the composite of two continuous functions between topological spaces is also continuous. The proof is particularly simple, illustrating that many familiar theorems for real functions can actually be proved more easily in the general setting of topological spaces.

Theorem 4.4 Composition Rule for continuous functions

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces. Let $f: X \to Y$ be $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous and let $g: Y \to Z$ be $(\mathcal{T}_Y, \mathcal{T}_Z)$-continuous. Then the composed function $g \circ f: X \to Z$ is $(\mathcal{T}_X, \mathcal{T}_Z)$-continuous.

Recall that $(g \circ f)(x) = g(f(x))$.

Proof Let $U \in \mathcal{T}_Z$. We must show that $(g \circ f)^{-1}(U) \in \mathcal{T}_X$.

By definition, $(g \circ f)^{-1}(U)$ is the set of all $x \in X$ such that $(g \circ f)(x) \in U$. Now, $(g \circ f)(x) \in U$ if and only if $f(x) \in g^{-1}(U)$, and this is so if and only if $x \in f^{-1}(g^{-1}(U))$. Thus,

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)).$$

Since g is continuous, $g^{-1}(U) \in \mathcal{T}_Y$.

Since f is continuous, $f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$.

Thus $g \circ f$ is continuous. ■

Next, we show that the continuity of a function $f: X \to Y$ carries over when the domain is replaced by a subset A of X with the subspace topology.

Theorem 4.5 Restriction Rule for continuous functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous. Let A be a subset of X and let $\mathcal{T}_A$ be the subspace topology inherited from $\mathcal{T}_X$. Then $f|_A : A \to Y$ is $(\mathcal{T}_A, \mathcal{T}_Y)$-continuous.

Recall that $f|_A$ denotes the *restriction* of f to the set A: $f|_A(x) = f(x)$, for each $x \in A$.

Proof Let $U \in \mathcal{T}_Y$. We must show that $f|_A^{-1}(U) \in \mathcal{T}_A$.

Now,

$$\begin{aligned} f|_A^{-1}(U) &= \{x \in A : f|_A(x) \in U\} = \{x \in A : f(x) \in U\} \\ &= A \cap \{x \in X : f(x) \in U\} = A \cap f^{-1}(U). \end{aligned}$$

Since f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous, we know that $f^{-1}(U) \in \mathcal{T}_X$. So, by the definition of the subspace topology $\mathcal{T}_A$, we have $A \cap f^{-1}(U) \in \mathcal{T}_A$, and hence $f|_A^{-1}(U) \in \mathcal{T}_A$.

Thus, $f|_A : A \to Y$ is $(\mathcal{T}_A, \mathcal{T}_Y)$-continuous. ■

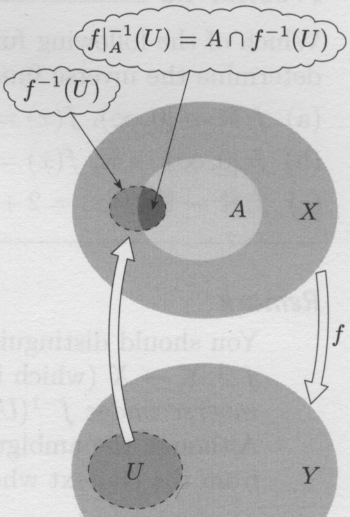

Figure 4.4

Finally, suppose that the values taken by a function $f: X \to Y$ are contained in a subset B of Y. In this case, we would expect continuity to carry over when the codomain is replaced by B, with the subspace topology. This is indeed the case.

Theorem 4.6

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous function. Let B be a subset of Y such that $f(X) \subseteq B$ and let $\mathcal{T}_B$ be the subspace topology on B inherited from $\mathcal{T}_Y$. Then $f: X \to B$ is $(\mathcal{T}_X, \mathcal{T}_B)$-continuous.

Problem 4.4

Prove Theorem 4.6.

4.2 Homeomorphisms

This subsection is concerned with an important class of continuous functions — those whose inverse functions are also continuous.

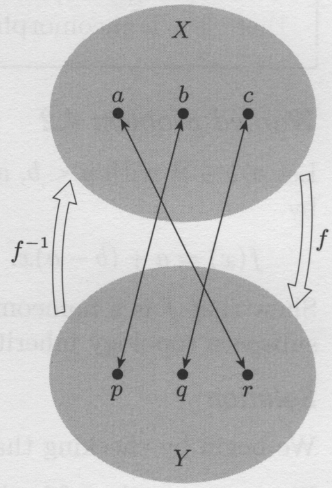

Figure 4.5

For a function $f: X \to Y$ to have an inverse function $f^{-1}: Y \to X$, we require that f is *onto* (so that f^{-1} is defined on the whole of Y) and *one–one* (so that $f^{-1}(y)$ takes exactly one value in X for each $y \in Y$). A function with both these properties is called a *bijection.*

Definition

A function $f: X \to Y$ is a **bijection** if f is both:

one–one: $f(x_1) = f(x_2)$ implies that $x_1 = x_2$, for all $x_1, x_2 \in X$;

onto: $f(X) = Y$.

In this case we also say that f is **bijective**.

A one–one function is also called an *injection.*
An onto function is also called a *surjection.*

Remark

If $f: X \to Y$ is a bijection then so is the inverse function $f^{-1}: Y \to X$.

You are asked to show this in the exercises for this unit.

Problem 4.5

Which of the following functions are bijective? For each bijective function, determine the inverse function.

(a) $f: \mathbb{R} \to [0, \infty)$, $f(x) = x^2$

(b) $f: [0, \infty) \to \mathbb{R}$, $f(x) = 1/(x+1)$

(c) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2 + 3x$

Remark

You should distinguish carefully between the *inverse function* $f^{-1}: Y \to X$ (which is defined *only* when f is a bijection) and the *inverse image* $f^{-1}(U)$ of a subset U of the codomain of *any* function. Although the ambiguity of notation is unfortunate, it is usually clear from the context what is meant.

We now look at the class of continuous bijections whose inverses are also continuous.

Definition

A **homeomorphism** between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ is a bijection $f: X \to Y$ such that f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous and f^{-1} is $(\mathcal{T}_Y, \mathcal{T}_X)$-continuous.

Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are **homeomorphic** if there is a homeomorphism between them.

A *homeo*morphism in topology is the equivalent of an *iso*morphism in algebra. It is not equivalent to a *homo*morphism, which need not be a bijection.

Worked problem 4.2

Let $a, b \in \mathbb{R}$ with $a < b$, and consider the function $f: (0,1) \to (a, b)$ defined by

$$f(x) = a + (b-a)x.$$

Show that f is a homeomorphism between $(0,1)$ and (a,b), each with the subspace topology inherited from the Euclidean topology.

Solution

We begin by checking that f is a bijection.

First we note that f is clearly a mapping from $(0,1)$ to (a,b).

Next, we show that f is one–one:

$$\begin{aligned} f(x_1) = f(x_2) &\Longrightarrow a + (b-a)x_1 = a + (b-a)x_2 \\ &\Longrightarrow (b-a)x_1 = (b-a)x_2 \\ &\Longrightarrow x_1 = x_2. \end{aligned}$$

We now show that f is onto. Let $y \in (a,b)$. Then

$$\begin{aligned} f(x) = y &\iff a + (b-a)x = y \\ &\iff x = (y-a)/(b-a). \end{aligned}$$

Since $0 < y - a < b - a$, it follows that $(y-a)/(b-a) \in (0,1)$.

Thus, $y = f((y-a)/(b-a))$, and so f is onto.

Since f is both one–one and onto, it is a bijection.

The inverse function $f^{-1}\colon (a,b) \to (0,1)$ is a bijection defined by $f^{-1}(y) = (y-a)/(b-a)$.

The functions $x \mapsto a + (b-a)x$ and $y \mapsto (y-a)/(b-a)$ are basic continuous functions on $\mathbb{R}$ with respect to the Euclidean distance function. Thus they are continuous with respect to the Euclidean topology on $\mathbb{R}$. Hence, by Theorems 4.5 and 4.6, f and f^{-1} are continuous with respect to the subspace topologies on $(0,1)$ and (a,b) inherited from the Euclidean topology on $\mathbb{R}$.

Therefore f is a homeomorphism between $(0,1)$ and (a,b). ■

The next problem exhibits a homeomorphism between a circle and an ellipse.

Problem 4.6

Let $(C, \mathcal{T}_C)$ be the unit circle $C = \{(x,y)\colon x^2 + y^2 = 1\}$ with the subspace topology inherited from the Euclidean topology on $\mathbb{R}^2$. Let $0 < a < b$ and let $(E, \mathcal{T}_E)$ be the ellipse $E = \{(x,y) : (x/a)^2 + (y/b)^2 = 1\}$ with the subspace topology inherited from the Euclidean topology on $\mathbb{R}^2$.

Show that the function $f\colon C \to E$, defined by $f(x,y) = (ax, by)$, is a homeomorphism between $(C, \mathcal{T}_C)$ and $(E, \mathcal{T}_E)$.

The definition of continuity for topological spaces gives the following useful characterization of a homeomorphism.

> ***Another definition of homeomorphism***
>
> Let $f\colon X \to Y$ be a bijection. Then f is a **homeomorphism** between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ if:
>
> ► for each $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$;
>
> ► for each $V \in \mathcal{T}_X$, $f(V) \in \mathcal{T}_Y$.

Remarks

(i) The first condition is a straightforward use of the definition of continuity. The second condition derives from the continuity of $f^{-1}\colon Y \to X$, which tells us that $(f^{-1})^{-1}(V) \in \mathcal{T}_Y$ whenever $V \in \mathcal{T}_X$. Now $y \in (f^{-1})^{-1}(V)$, the inverse image set of V under the function f^{-1}, if and only if $f^{-1}(y) \in V$ and, since f is a bijection, this is true if and only if $y \in f(V)$. Hence we can replace $(f^{-1})^{-1}(V)$ by $f(V)$ to give us the second condition.

(ii) This alternative definition tells us that a homeomorphism between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ exists if and only if there is a one–one correspondence between the sets in $\mathcal{T}_X$ and the sets in $\mathcal{T}_Y$. We say that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are *topologically equivalent*.

You may have come across the expression: *a topologist is someone who doesn't know the difference between a doughnut and a teacup.* The reason for this remark is that these two figures are sets of points in Euclidean space that can be transformed into each other by homeomorphisms, and so are topologically equivalent.

This idea of transforming figures one into another is illustrated on the DVD and is explored further in Block B.

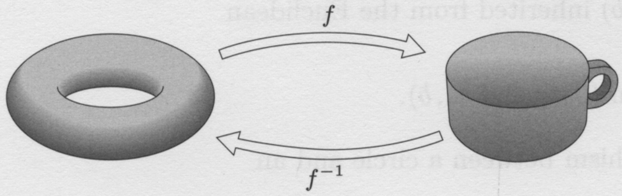

Figure 4.6

Let us now see how we can make use of this alternative definition.

Worked problem 4.3

Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$, and let

$$\mathcal{T}_X = \{\varnothing, \{a\}, \{a, b\}, X\}, \quad \mathcal{T}_Y = \{\varnothing, \{q\}, \{q, r\}, Y\}.$$

Show that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic.

You saw earlier that $\mathcal{T}_X$ and $\mathcal{T}_Y$ define topologies on X and Y, respectively.

Solution

We must find a bijection from X to Y that gives a one–one correspondence between the sets in $\mathcal{T}_X$ and the sets in $\mathcal{T}_Y$.

We define $f: X \to Y$ by $f(a) = q$, $f(b) = r$ and $f(c) = p$. This is a bijection, and gives a one–one correspondence between the open sets as shown.

$$\begin{array}{ccccc} \mathcal{T}_X & \varnothing & \{a\} & \{a,b\} & X \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathcal{T}_Y & \varnothing & \{q\} & \{q,r\} & Y \end{array}$$

■

Problem 4.7

Let $X = \{a, b, c\}$, $Y = \{p, q, r\}$, and let

$$\mathcal{T}_X = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}, \quad \mathcal{T}_Y = \{\varnothing, \{p\}, \{r\}, \{p, r\}, Y\}.$$

Show that $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are homeomorphic.

You may assume that $\mathcal{T}_X$ and $\mathcal{T}_Y$ are topologies on X and Y, respectively.

We conclude this section with two results that generalize the Restriction and Composition Rules from continuous functions to homeomorphisms. In each case, we first need a corresponding result for bijections.

First, we prove the Restriction Rule for bijections.

> **Theorem 4.7 Restriction Rule for bijections**
>
> Let $f: X \to Y$ be a bijection. Let A be a subset of X and let $f(A) = B$. Then $f|_A : A \to B$ is a bijection.

Proof By the definition of B, f maps A onto B. Also f is one–one, since it is a bijection. Therefore $f|_A$ is a bijection from A to B. ■

We can now prove the Restriction Rule for homeomorphisms.

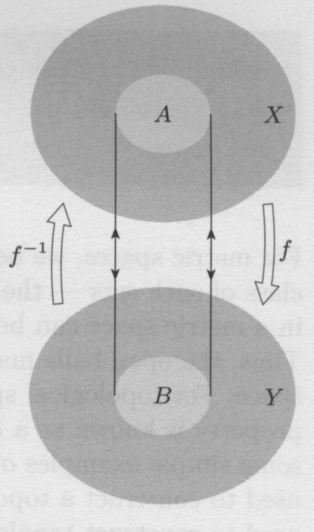

Figure 4.7

> ***Theorem 4.8 Restriction Rule for homeomorphisms***
>
> Let $f: X \to Y$ be a homeomorphism between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Let A be a subset of X and let $f(A) = B$. If $\mathcal{T}_A$ and $\mathcal{T}_B$ are the subspace topologies inherited from $\mathcal{T}_X$ and $\mathcal{T}_Y$, respectively, then $f|_A: A \to B$ is a homeomorphism between $(A, \mathcal{T}_A)$ and $(B, \mathcal{T}_B)$.

Proof Since f is a homeomorphism, it is a bijection. Therefore, by Theorem 4.7, $f|_A: A \to B$ is a bijection.

Since f is a homeomorphism between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$, f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous. Hence, by Theorems 4.5 and 4.6, $f|_A: A \to B$ is $(\mathcal{T}_A, \mathcal{T}_B)$-continuous. Similarly, $f|_A^{-1}: B \to A$ is $(\mathcal{T}_B, \mathcal{T}_A)$-continuous.

Hence $f|_A: A \to B$ is a homeomorphism between $(A, \mathcal{T}_A)$ and $(B, \mathcal{T}_B)$. ■

To generalize the Composition Rule to homeomorphisms, we first need the Composition Rule for bijections.

> ***Theorem 4.9 Composition Rule for bijections***
>
> Let $f: X \to Y$ and $g: Y \to Z$ be bijections. Then $g \circ f$ is a bijection from X to Z.

Proof We first show that $g \circ f$ is one–one.

Suppose that $(g \circ f)(x_1) = (g \circ f)(x_2)$: that is, $g(f(x_1)) = g(f(x_2))$. Since g is one–one, $f(x_1) = f(x_2)$. Then, since f is one–one, $x_1 = x_2$. Thus $g \circ f$ is one–one.

We next show that $g \circ f$ is onto.

Since f and g are both onto, $f(X) = Y$ and $g(Y) = Z$. So, $(g \circ f)(X) = g(f(X)) = g(Y) = Z$. Thus $g \circ f$ is onto.

Thus $g \circ f$ is a bijection. ■

We can now prove the Composition Rule for homeomorphisms.

> ***Theorem 4.10 Composition Rule for homeomorphisms***
>
> Let $f: X \to Y$ be a homeomorphism between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ and let $g: Y \to Z$ be a homeomorphism between $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$. Then $g \circ f$ is a homeomorphism between $(X, \mathcal{T}_X)$ and $(Z, \mathcal{T}_Z)$.

Proof Since f and g are homeomorphisms, they are bijections. It follows from Theorem 4.9 that $g \circ f$ is a bijection.

Since f and g are homeomorphisms, f, g, f^{-1} and g^{-1} are all continuous. Therefore, by Theorem 4.4, $g \circ f$ is $(\mathcal{T}_X, \mathcal{T}_Z)$-continuous and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is $(\mathcal{T}_Z, \mathcal{T}_X)$-continuous.

Thus $g \circ f$ is a homeomorphism between $(X, \mathcal{T}_X)$ and $(Z, \mathcal{T}_Z)$. ■

5 Bases

After working through this section, you should be able to:

- explain and use the concept of a *base* for a topology;
- explain and use the *product topology*.

For metric spaces, we began our study of open sets by considering a special class of such sets — the open balls. In fact, as we shall see, every open set in a metric space can be expressed as the union of a family of open balls. Thus, the open balls may be considered as the 'building blocks' of a metric space. For topological spaces in general, a collection of open sets with this property is known as a *base* for the topology. In this section, we look at some simple examples of bases and show how the idea of a base can be used to construct a topology on a set. We shall also see how bases can be used to construct topologies on product sets.

5.1 Examples of bases

We begin this subsection by defining a *base* for a topology.

Definition

Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B}$ of subsets of X is a **base** for the topology $\mathcal{T}$ if:

(B1) $\mathcal{B} \subseteq \mathcal{T}$;

(B2) each open set $U \in \mathcal{T}$ is the union of a family of sets in $\mathcal{B}$.

If we add open sets to a base, we still have a base, as we now ask you to prove.

Problem 5.1

Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{B}$ be a base for $\mathcal{T}$ and let $\mathcal{B} \subseteq \mathcal{C} \subseteq \mathcal{T}$. Show that $\mathcal{C}$ is a base for $\mathcal{T}$.

We now prove that the open balls of a metric space form a base for that space.

Theorem 5.1

Let d be a metric on a set X. Then the set of d-open balls is a base for the topology $\mathcal{T}(d)$.

Recall that $\mathcal{T}(d)$ is the family of all the d-open subsets of X. We observed in Section 1 that any metric space (X, d) gives rise to a topological space $(X, \mathcal{T}(d))$.

Proof We must show that (B1) and (B2) are satisfied.

(B1) Each d-open ball is a d-open set, and therefore belongs to $\mathcal{T}(d)$. Thus (B1) is satisfied.

(B2) Let $U \in \mathcal{T}(d)$. We must show that U can be written as a union of d-open balls. To do this, we use the definition of an open set in a metric space: this tells us that, for each $u \in U$, there exists a d-open ball $B_d(u, r)$ such that $u \in B_d(u, r) \subseteq U$. Let us write this ball as $B(u)$. Since $u \in B(u)$ for each $u \in U$, it follows that

Unit A2, Section 4.

Remember that the radius r of $B(u)$ depends on u.

$$U \subseteq \bigcup_{u \in U} B(u).$$

Moreover, since $B(u) \subseteq U$ for each $u \in U$, we may conclude that

$$\bigcup_{u \in U} B(u) \subseteq U.$$

Combining these results, we have

$$U = \bigcup_{u \in U} B(u).$$

Thus (B2) is satisfied.

Since (B1) and (B2) are satisfied, the d-open balls form a base for the topology $\mathcal{T}(d)$. ■

Remark

Theorem 5.1 tells us, for example, that the open intervals form a base for the Euclidean topology on $\mathbb{R}$ and that the open discs form a base for the Euclidean topology on $\mathbb{R}^2$.

There can be many different bases for the same topology. We have just remarked that the open discs form a base for the Euclidean topology on $\mathbb{R}^2$. We now ask you to show that the open squares also form a base for the Euclidean topology on $\mathbb{R}^2$.

You met the idea of an *open rectangle* in *Unit A2*, Section 4. It is a set of the form $(a, b) \times (c, d)$. An *open square* is an open rectangle $(a, b) \times (c, d)$ with $b - a = d - c$.

Problem 5.2

Show that the family of open squares in $\mathbb{R}^2$ is a base for the Euclidean topology on $\mathbb{R}^2$.

Hint In the Euclidean topology on $\mathbb{R}^2$, the open ball $B(\mathbf{u})$ of radius $r(\mathbf{u})$ centred at a point $\mathbf{u}$ contains the open square $S(\mathbf{u})$ of side length $r(\mathbf{u})$ also centred at $\mathbf{u}$ (see Figure 5.1).

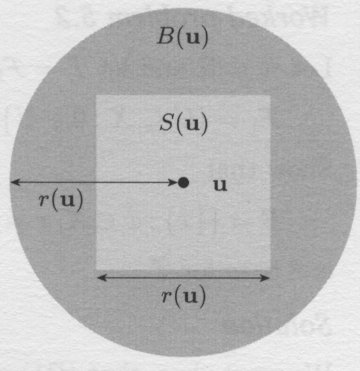

Figure 5.1

Bases may be finite, countably infinite or uncountable. For example, if a topology $\mathcal{T}$ is finite then, by definition, a base for $\mathcal{T}$ must be finite. The open intervals form an uncountable base for the Euclidean topology on $\mathbb{R}$, as do the open discs and open squares for the Euclidean topology on $\mathbb{R}^2$. The next worked problem provides an example of a countably infinite base.

Worked problem 5.1

Show that the family of open intervals with rational endpoints is a base for the Euclidean topology on $\mathbb{R}$.

Solution

We must show that (B1) and (B2) are satisfied.

(B1) Each open interval with rational endpoints is open with respect to the Euclidean metric on $\mathbb{R}$, and therefore belongs to the Euclidean topology on $\mathbb{R}$. Thus (B1) is satisfied.

(B2) Let U be open with respect to the Euclidean metric on $\mathbb{R}$. We must show that U can be written as the union of open intervals with rational endpoints.

We know, from the definition of an open set in a metric space, that each point $a \in U$ is the centre of an open ball (an open interval, in this context) entirely contained in U. Let this interval be $(a-x, a+x)$, where $x > 0$.

Now the rational numbers have the property that, between any two *real* numbers y and z with $y < z$, there is a *rational* number r with $y < r < z$. It follows that there are a rational number r_a such that $a - x < r_a < a$ and a rational number s_a such that $a < s_a < a + x$.

This is known as the *density property* of the rational numbers with respect to the real numbers.

For each $a \in U$, let I_a be the open interval (r_a, s_a). Then, since $a \in I_a$,

$$U \subseteq \bigcup_{a \in U} I_a,$$

and, since $I_a \subseteq (a-x, a+x) \subseteq U$, for each $a \in U$,

$$\bigcup_{a \in U} I_a \subseteq U.$$

Combining these, we have

$$U = \bigcup_{a \in U} I_a.$$

Thus (B2) is satisfied.

Since (B1) and (B2) are satisfied, the family of open intervals with rational endpoints forms a base for the Euclidean topology on $\mathbb{R}$. ■

Let us now identify a base for each of the either-or topologies that we considered in Section 3.

Worked problem 5.2

Let $X = \mathbb{R}$ and let $\mathcal{T} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{U \subseteq X : 0 \notin U\} \quad \text{and} \quad \mathcal{F}_2 = \{U \subseteq X : 0 \in U \text{ and } U^c \text{ is finite}\}.$$

You showed in Problem 3.6 that $\mathcal{T}$ is a topology on $\mathbb{R}$.

Show that

$$\mathcal{B} = \{\{x\} : x \in X, x \neq 0\} \cup \mathcal{F}_2$$

is a base for $\mathcal{T}$.

Solution

We must show that (B1) and (B2) are satisfied.

(B1) If $x \in X$ and $x \neq 0$, then $\{x\} \in \mathcal{F}_1 \subseteq \mathcal{T}$. Also, $\mathcal{F}_2 \subseteq \mathcal{T}$.

Thus (B1) is satisfied.

(B2) Let $U \in \mathcal{T}$. We must show that U can be written as a union of sets in $\mathcal{B}$.

If $U \in \mathcal{F}_1$, we can write

$$U = \bigcup_{x \in U} \{x\}.$$

Since $0 \notin U$, $\{x\} \in \mathcal{B}$ for each $x \in U$.

If $U \in \mathcal{F}_2$, then $U \in \mathcal{B}$, and so there is nothing to show.

Thus (B2) is satisfied.

Since (B1) and (B2) are satisfied, $\mathcal{B}$ is a base for $\mathcal{T}$. ■

Problem 5.3

Let $X = [-1, 1]$ and let $\mathcal{T} = \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{U \subseteq X : 0 \notin U\} \quad \text{and} \quad \mathcal{F}_2 = \{U \subseteq X : (-1, 1) \subseteq U\}.$$

We showed in Worked problem 3.3 that $\mathcal{T}$ is a topology on X.

Show that

$$\mathcal{B} = \{\{x\} : x \in X, x \neq 0\} \cup \{(-1, 1)\}$$

is a base for $\mathcal{T}$.

We conclude this subsection by showing that, given a base for a topology on a set X, the construction of a corresponding base for the subspace topology on a subset A of X is what we would expect: we take the intersection of each set of the base with A.

Theorem 5.2

Let $(X, \mathcal{T})$ be a topological space, let $(A, \mathcal{T}_A)$ be a subspace of this space and let $\mathcal{B}$ be a base for $\mathcal{T}$. Then

$$\mathcal{B}_A = \{B \cap A : B \in \mathcal{B}\}$$

is a base for $\mathcal{T}_A$.

Proof We show that (B1) and (B2) are satisfied.

(B1) It follows from the definitions of $\mathcal{B}$ and $\mathcal{T}_A$ that $\mathcal{B}_A \subseteq \mathcal{T}_A$, and so (B1) is satisfied.

(B2) Let $U \in \mathcal{T}_A$. We must show that U can be written as the union of a family of sets in $\mathcal{B}_A$.

First we note that there exists $V \in \mathcal{T}$ such that $U = V \cap A$. Since $\mathcal{B}$ is a base for $\mathcal{T}$, there is a family of sets $\{B_i : i \in I\}$ in $\mathcal{B}$ such that

$$V = \bigcup_{i \in I} B_i.$$

Thus, by the first Distributive Law for families of sets,

Theorem 2.6.

$$U = V \cap A = \left(\bigcup_{i \in I} B_i\right) \cap A = \bigcup_{i \in I} (B_i \cap A),$$

so that U is a union of sets from $\mathcal{B}_A$.

Thus (B2) is satisfied.

Since (B1) and (B2) are satisfied, $\mathcal{B}_A$ is a base for $\mathcal{T}_A$. ■

Earlier in this subsection we saw that the open intervals in $\mathbb{R}$ form a base for the Euclidean topology on $\mathbb{R}$. It follows from Theorem 5.2 that the open intervals in $(0,\infty)$ form a base for the subspace topology on $(0,\infty)$ inherited from the Euclidean topology on $\mathbb{R}$.

5.2 Bases and continuity

Bases can sometimes be used to simplify the checking of continuity. Up to now, when proving continuity, we have had to consider *all the open sets* in the codomain topology. We now show that we need consider *only the open sets in a base* for the codomain topology.

Theorem 5.3

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{B}$ be a base for $\mathcal{T}_Y$. A function $f\colon X \to Y$ is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous if $f^{-1}(B) \in \mathcal{T}_X$ whenever $B \in \mathcal{B}$.

Proof Suppose that $f^{-1}(B) \in \mathcal{T}_X$ whenever $B \in \mathcal{B}$. By the definition of continuity, we must show that $f^{-1}(U) \in \mathcal{T}_X$ whenever $U \in \mathcal{T}_Y$.

To do this, we take $U \in \mathcal{T}_Y$ and use (B2) to write

$$U = \bigcup_{i\in I} B_i,$$

where $B_i \in \mathcal{B}$ for each $i \in I$. Using the Inverse Mapping Rule for functions of unions, we obtain

Theorem 2.7.

$$f^{-1}(U) = f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i).$$

We know that $f^{-1}(B_i) \in \mathcal{T}_X$, for each $i \in I$. So, by (T3),

$$f^{-1}(U) = \bigcup_{i\in I} f^{-1}(B_i) \in \mathcal{T}_X.$$

Thus, f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous. ■

Worked problem 5.3

Let $\mathcal{T}$ be the subspace topology on $(0,\infty)$ inherited from the Euclidean topology on $\mathbb{R}$ and let $\mathcal{B}$ denote the base for $\mathcal{T}$ consisting of all the open intervals in $(0,\infty)$. Use Theorem 5.3 to show that the following function is $(\mathcal{T}, \mathcal{T})$-continuous:

$$f\colon (0,\infty) \to (0,\infty) \quad \text{given by} \quad f(x) = 1/x.$$

Solution

Let $(a,b) \in \mathcal{B}$. We must show that $f^{-1}((a,b)) \in \mathcal{T}$.

Now, for any $y \in (a,b)$, $f^{-1}(\{y\}) = \{1/y\}$, and hence, since f is a decreasing function,

$$f^{-1}((a,b)) = (1/b, 1/a) \in \mathcal{T}.$$

It follows from Theorem 5.3 that f is $(\mathcal{T}, \mathcal{T})$-continuous. ■

Problem 5.4

Let $\mathcal{T}$ be the subspace topology on $(0, \infty)$ inherited from the Euclidean topology on $\mathbb{R}$ and let $\mathcal{B}$ denote the base for $\mathcal{T}$ consisting of all the open intervals in $(0, \infty)$. Use Theorem 5.3 to show that the following function is $(\mathcal{T}, \mathcal{T})$-continuous:

$$f\colon (0,\infty) \to (0,\infty) \quad \text{given by} \quad f(x) = x^2.$$

5.3 Constructing topologies

Earlier in this section, we took some topologies and tried to find bases for these. We now look at the converse problem — can we use the idea of a base to construct a topology on a set?

If $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on a set X, then (by definition) each $U \in \mathcal{T}$ is the union of a family of sets in $\mathcal{B}$. Conversely (since each set of $\mathcal{B}$ is in $\mathcal{T}$), every union of sets in $\mathcal{B}$ is in $\mathcal{T}$. Thus, it is tempting to suppose that *any* family $\mathcal{B}$ of subsets of a set X defines a topology on X, consisting of all unions of families of sets in $\mathcal{B}$. However, this is not so, for two reasons:

- it may be impossible to express X as a union of sets in $\mathcal{B}$;
- the intersection of two sets in $\mathcal{B}$ may not be a union of sets in $\mathcal{B}$.

But if $\mathcal{B}$ does *not* fail on either of these counts, then the unions of families of sets of $\mathcal{B}$ *do* form a topology on X. We now prove this.

Theorem 5.4

Let X be a set and let $\mathcal{B}$ be a family of subsets of X satisfying:

(B3) $X = \bigcup_{B\in\mathcal{B}} B$;

(B4) if $B_1, B_2 \in \mathcal{B}$ then $B_1 \cap B_2 \in \mathcal{B}$.

Let $\mathcal{T}$ be the family of all unions of families of sets in $\mathcal{B}$:

$$\mathcal{T} = \left\{ \bigcup_{i\in I} B_i \colon \{B_i : i \in I\} \text{ is a family of sets in } \mathcal{B} \right\}.$$

Then $\mathcal{T}$ is a topology on X, and $\mathcal{B}$ is a base for $\mathcal{T}$.

Remarks

(i) By a 'family of sets in $\mathcal{B}$', we mean a subfamily of $\mathcal{B}$ — that is, a family of sets each of which individually belongs to $\mathcal{B}$.

(ii) As we shall see, the reason for (B3) is to ensure that $\mathcal{T}$ satisfies (T1) and the reason for (B4) is to ensure that $\mathcal{T}$ satisfies (T2). The way in which $\mathcal{T}$ is defined then ensures that $\mathcal{T}$ satisfies (T3).

Proof We prove first that $\mathcal{T}$ is a topology on X.

(T1) The empty set $\varnothing$ is the union of an empty collection of sets in $\mathcal{B}$ and thus belongs to $\mathcal{T}$. Also, (B3) guarantees that $X \in \mathcal{T}$. Thus (T1) is satisfied.

(T2) Let $U_1, U_2 \in \mathcal{T}$ and let $U = U_1 \cap U_2$; we must show that $U \in \mathcal{T}$.

From the definition of $\mathcal{T}$, there are families $\{B_i : i \in I\}$ and $\{B_j : j \in J\}$ of sets in $\mathcal{B}$ such that

$$U_1 = \bigcup_{i \in I} B_i \quad \text{and} \quad U_2 = \bigcup_{j \in J} B_j.$$

Hence, using the first Distributive Law for families of sets, Theorem 2.6.

$$U = \left(\bigcup_{i \in I} B_i\right) \cap \left(\bigcup_{j \in J} B_j\right) = \bigcup_{i \in I, j \in J} (B_i \cap B_j).$$

Since, by (B4), each $B_i \cap B_j \in \mathcal{B}$, U is a union of sets from $\mathcal{B}$, and so $U \in \mathcal{T}$.

Thus (T2) is satisfied.

(T3) Let $\{U_i : i \in I\}$ be a collection of sets in $\mathcal{T}$ and let $U = \bigcup_{i \in I} U_i$. We must show that $U \in \mathcal{T}$.

For each $i \in I$, there is a family of sets in $\mathcal{B}$ whose union is U_i. The collection of all these families is itself a family of sets in $\mathcal{B}$, whose union is the set U, and thus U belongs to $\mathcal{T}$.

Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}$ is a topology on X.

From the definition of $\mathcal{T}$, (B1) and (B2) are clearly satisfied, and so $\mathcal{B}$ is a base for $\mathcal{T}$. ■

Theorem 5.4 provides a useful method for constructing topologies, since it is often much simpler to find a collection of sets satisfying (B3) and (B4) than it is to find a collection of sets satisfying (T1)–(T3). The following worked problem shows how we can use Theorem 5.4 to construct an alternative topology on $\mathbb{R}$.

Worked problem 5.4

Let $\mathcal{B} = \{\varnothing\} \cup \{[a, b] : a, b \in \mathbb{Z} \text{ and } a \leq b\}$.

Show that $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on $\mathbb{R}$, and define $\mathcal{T}$.

Solution

We show that (B3) and (B4) are satisfied.

(B3) Each set in $\mathcal{B}$ is a subset of $\mathbb{R}$, so $\bigcup_{B \in \mathcal{B}} B \subseteq \mathbb{R}$. Also

$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} [-n, n] \subseteq \bigcup_{B \in \mathcal{B}} B.$$

Hence $\mathbb{R} = \bigcup_{B \in \mathcal{B}} B$.

(B4) Let $B_1 = [a, b]$ and $B_2 = [c, d]$ belong to $\mathcal{B}$. We can assume, without loss of generality, that $a \le c$. We must show that $B_1 \cap B_2 \in \mathcal{B}$.

If $b < c$, then $B_1 \cap B_2 = \varnothing$.

If $b = c$, then $B_1 \cap B_2 = \{b\} = [b, b]$.

If $b > c$, then (since $a \le c$) $B_1 \cap B_2 = [c, e]$, where $e = \min\{b, d\}$.

So, in each case, $B_1 \cap B_2 \in \mathcal{B}$.

Since (B3) and (B4) are satisfied, it follows from Theorem 5.4 that $\mathcal{B}$ is a base for a topology on $\mathbb{R}$, given by

$$\mathcal{T} = \left\{ \bigcup_{i \in I} B_i : \{B_i : i \in I\} \text{ is a family of sets in } \mathcal{B} \right\}. \qquad \blacksquare$$

The topology that we have just constructed on $\mathbb{R}$ is certainly different from the Euclidean topology on $\mathbb{R}$, since it contains closed intervals and these do not belong to the Euclidean topology on $\mathbb{R}$.

The next problem shows how Theorem 5.4 can be used to construct a topology on a set with finitely many elements.

Problem 5.5

Let $X = \{a, b, c\}$ and let $\mathcal{B} = \{\varnothing, \{a\}, \{c\}, \{a, b\}\}$.

Show that $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on X, and list all the sets in $\mathcal{T}$.

5.4 The product topology

In this subsection we show how Theorem 5.4 can be used to construct a topology on the *product* of two sets.

In *Unit A2* you saw that, given two *metric* spaces (X_1, d_1) and (X_2, d_2), there are several different ways in which the metrics d_1 and d_2 can be combined to form a metric on the product set $X_1 \times X_2$. Moreover, as we discovered at the end of Subsection 4.1 of *Unit A2*, each of the ways we described gives rise to the *same* open sets — that is, to the same *product topology.*

Thus, suppose that we have two *topological* spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$. It seems reasonable to hope that we can form a topology $\mathcal{T}$ on $X_1 \times X_2$ from $\mathcal{T}_1$ and $\mathcal{T}_2$. Unfortunately, $\mathcal{T}$ cannot be defined simply as the collection of sets of the form $U_1 \times U_2$, where $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$, since the unions of such sets are not usually of this form. For example, suppose that we wish to form a topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ using the Euclidean topology on $\mathbb{R}$. We know that the open intervals all belong to the Euclidean topology on $\mathbb{R}$. Taking the product of two open intervals, we obtain an open rectangle. When we take the union of two open rectangles, however, we do not usually obtain an open rectangle.

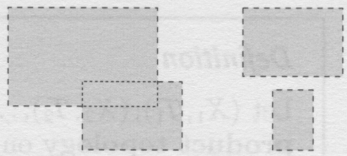

Figure 5.2 Possible unions of two open rectangles

We have, however, seen that the family of open squares forms a *base* for the Euclidean topology on $\mathbb{R}^2$. Since the family of open rectangles contains the open squares as a subfamily, the open rectangles also form a base for the Euclidean topology. We now show that the family of products of sets in the topologies on two sets *always gives a base for a topology on the product set.*

See Problem 5.2.

See Problem 5.1.

Theorem 5.5

Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces. Then the family

$$\mathcal{B} = \{U_1 \times U_2 : U_1 \in \mathcal{T}_1,\ U_2 \in \mathcal{T}_2\}$$

is a base for a topology on $X = X_1 \times X_2$.

Proof We show that $\mathcal{B}$ has properties (B3) and (B4); the result then follows from Theorem 5.4.

(B3) For each $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$, $U_1 \times U_2 \subseteq X_1 \times X_2 = X$, so that

$$\bigcup_{B \in \mathcal{B}} B \subseteq X.$$

Furthermore, since $X_1 \in \mathcal{T}_1$ and $X_2 \in \mathcal{T}_2$, it follows that $X = X_1 \times X_2 \in \mathcal{B}$, so that

$$X \subseteq \bigcup_{B \in \mathcal{B}} B.$$

Thus

$$X = \bigcup_{B \in \mathcal{B}} B$$

and so (B3) is satisfied.

(B4) Let $A, B \in \mathcal{B}$ and let $C = A \cap B$. We must show that $C \in \mathcal{B}$.

By the definition of $\mathcal{B}$, there are sets $U_1, V_1 \in \mathcal{T}_1$ and $U_2, V_2 \in \mathcal{T}_2$ such that

$$A = U_1 \times U_2 \quad \text{and} \quad B = V_1 \times V_2.$$

Then, by the Distributive Law of intersections over products, Theorem 2.9.

$$C = (U_1 \times U_2) \cap (V_1 \times V_2) = (U_1 \cap V_1) \times (U_2 \cap V_2).$$

Since $U_1 \cap V_1 \in \mathcal{T}_1$ and $U_2 \cap V_2 \in \mathcal{T}_2$, it follows that $C \in \mathcal{B}$. Thus (B4) is satisfied. ■

The topology obtained in this way is called the *product topology*. By adding one space at a time, we can build up the corresponding result for any finite number of spaces. This leads to the natural definition of the product topology on $X_1 \times X_2 \times \cdots \times X_n$.

Definition

Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), \ldots, (X_n, \mathcal{T}_n)$ be topological spaces. The **product topology** on $X_1 \times X_2 \times \cdots \times X_n$ is the topology with base

$$\mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_n : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2, \ldots, U_n \in \mathcal{T}_n\}.$$

We conclude this section by looking at functions defined on product spaces.

Recall, from *Unit A2*, that, given two sets X_1 and X_2 and the product set $X = X_1 \times X_2$, the projection functions $p_1 \colon X \to X_1$ and $p_2 \colon X \to X_2$ are given by $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$. We saw that these functions are continuous in the context of metric spaces. They are also continuous in the more general context of topological spaces.

Unit A2, Theorem 3.6.

> **Theorem 5.6**
>
> Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and let $\mathcal{T}$ be the product topology on $X_1 \times X_2$. Then the projection functions
>
> $$p_1 : X_1 \times X_2 \to X_1 \quad \text{and} \quad p_2 : X_1 \times X_2 \to X_2$$
>
> are respectively $(\mathcal{T}, \mathcal{T}_1)$-continuous and $(\mathcal{T}, \mathcal{T}_2)$-continuous.

Proof We prove that p_1 is $(\mathcal{T}, \mathcal{T}_1)$-continuous; the proof for p_2 is essentially the same.

Let $U \in \mathcal{T}_1$. We must show that $p_1^{-1}(U) \in \mathcal{T}$. Now

$$p_1^{-1}(U) = \{(x_1, x_2) : x_1 \in U\} = U \times X_2.$$

Since $U \in \mathcal{T}_1$ and $X_2 \in \mathcal{T}_2$, it follows that $U \times X_2 \in \mathcal{T}$; that is, $p_1^{-1}(U) \in \mathcal{T}$.

Thus, p_1 is $(\mathcal{T}, \mathcal{T}_1)$-continuous. ■

We saw in *Unit A2* that we can use projection functions to help prove the continuity of functions between metric spaces when the domain is a product set. We can generalize this technique to topological spaces. It relies on the following generalization of Theorem 3.7 of *Unit A2*.

> **Theorem 5.7**
>
> Let $(Y, \mathcal{T}_Y)$, $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be topological spaces and let $\mathcal{T}$ be the product topology on $X_1 \times X_2$. Then a function $f : Y \to X_1 \times X_2$ is $(\mathcal{T}_Y, \mathcal{T})$-continuous if and only if $p_1 \circ f : Y \to X_1$ is $(\mathcal{T}_Y, \mathcal{T}_1)$-continuous and $p_2 \circ f : Y \to X_2$ is $(\mathcal{T}_Y, \mathcal{T}_2)$-continuous.

Problem 5.6

Prove Theorem 5.7.

Hint Use Theorem 5.3.

As in the case of metric spaces, Theorems 5.6 and 5.7 can be generalized to products of n sets.

6 Comparing topologies

After working through this section, you should be able to:

- explain what it means for one topology to be *finer* or *coarser* than another;
- explain what are meant by *topologically equivalent* metrics;
- show that certain metrics are topologically equivalent.

In this section we shall see how we can compare topologies and metrics. In Subsection 6.1, we look at some of the consequences of having fewer (or more) open sets in a topology. In Subsection 6.2, we develop a simple technique that can be used to show that two metrics are equivalent, topologically speaking.

6.1 Comparable topologies on a set

You have seen that many different topologies can be defined on the same set. Given two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X, it may be that every set in $\mathcal{T}_1$ belongs to $\mathcal{T}_2$ — that is, $\mathcal{T}_1 \subseteq \mathcal{T}_2$. For example, this is the case when $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\varnothing, \{a, b\}, X\}$ and $\mathcal{T}_2 = \{\varnothing, \{b\}, \{a, b\}, \{b, c\}, X\}$. In contrast, there may be sets in $\mathcal{T}_1$ that are not in $\mathcal{T}_2$, and sets in $\mathcal{T}_2$ that are not in $\mathcal{T}_1$ — that is, neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ is a subset of the other. For example, this is the case when $X = \{a, b, c\}$, $\mathcal{T}_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{T}_2 = \{\varnothing, \{b\}, \{c\}, \{b, c\}, X\}$. We need some terminology to describe these possibilities.

Definition

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X.

If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or $\mathcal{T}_2 \subseteq \mathcal{T}_1$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are **comparable** topologies. Otherwise they are **not comparable**.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are comparable with $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_2$ is **finer** than $\mathcal{T}_1$ and $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$.

Remarks

(i) To understand why we use the terms *finer* and *coarser* in this context, we liken a topology to a fishing net. The open sets are analogous to the holes of the net — the more open sets there are, the finer the topology. For example, in Figure 6.1, $\mathcal{T}_2$ is a finer topology than $\mathcal{T}_1$.

(ii) The term *larger* is sometimes used instead of *finer*, and *smaller* instead of *coarser*.

(iii) Every topology on a set is a subset of the discrete topology on that set. Hence the discrete topology is comparable with every other topology on the set, and is finer than every other topology. Analogously, the indiscrete topology on a set is a subset of every other topology on that set. Hence the indiscrete topology is comparable with every other topology on the set, and is coarser than every other topology. The discrete topology is the *finest*, and the indiscrete the *coarsest*, topology on a set.

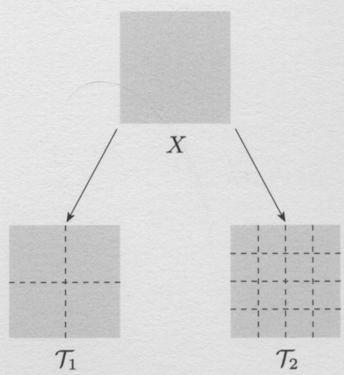

Figure 6.1

Problem 6.1

Let $X = \{a, b, c\}$, and let

$$\mathcal{T}_1 = \{\varnothing, \{b\}, X\}, \quad \mathcal{T}_2 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}, \quad \mathcal{T}_3 = \{\varnothing, \{a\}, X\}$$

be topologies on X. Which pairs of these topologies are comparable? For those pairs that are comparable, which topology is the finer of the pair?

You may assume that $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are topologies on X.

In Theorem 4.2 we showed that, if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set X, then the identity function on X is $(\mathcal{T}_1, \mathcal{T}_2)$-continuous if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$. We can now rewrite this result as follows.

Theorem 6.1

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X. The identity function on X is $(\mathcal{T}_1, \mathcal{T}_2)$-continuous if and only if $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

Problem 6.2

Let $X = \{a, b, c\}$ and let

$$\mathcal{T}_1 = \{\varnothing, \{b\}, X\}, \quad \mathcal{T}_2 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}, \quad \mathcal{T}_3 = \{\varnothing, \{a\}, X\}$$

be topologies on X. For which pairs of topologies is the identity function on X $(\mathcal{T}_i, \mathcal{T}_j)$-continuous?

You have seen that changing the topologies on the domain and codomain of a function can change the continuity of the function. We now prove a result that shows how continuity is affected by replacing a topology by a finer or coarser one.

Theorem 6.2

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f\colon X \to Y$ be continuous. The function f remains continuous if $\mathcal{T}_X$ is replaced by a finer topology and/or $\mathcal{T}_Y$ is replaced by a coarser topology.

Proof First suppose that $\mathcal{T}_1$ is a finer topology than $\mathcal{T}_X$.

If $U \in \mathcal{T}_Y$ then, since f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous, $f^{-1}(U) \in \mathcal{T}_X \subseteq \mathcal{T}_1$. Thus f is $(\mathcal{T}_1, \mathcal{T}_Y)$-continuous.

Now suppose that $\mathcal{T}_2$ is a coarser topology than $\mathcal{T}_Y$.

If $U \in \mathcal{T}_2$ then, since $\mathcal{T}_2 \subseteq \mathcal{T}_Y$, we have $U \in \mathcal{T}_Y$. Then, since f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous, $f^{-1}(U) \in \mathcal{T}_X$. Thus f is $(\mathcal{T}_X, \mathcal{T}_2)$-continuous. ■

Remark

We can deduce from Theorem 6.2 that the finer the topology on X, and/or the coarser the topology on Y, the more continuous functions there are from X to Y.

6.2 Comparing metrics

We now compare the topologies on a set that are determined by different metrics. Recall that a metric d on a set X determines a topology $\mathcal{T}(d)$ whose elements are the d-open subsets of X. If two metrics determine the same topology, we say that they are *topologically equivalent.*

Definition

Let d and e be two metrics on a set X. The metrics d and e are **topologically equivalent** if $\mathcal{T}(d) = \mathcal{T}(e)$.

There is a simple condition that ensures that two metrics are topologically equivalent. In order to establish this condition, we use the fact that $\mathcal{T}(d) = \mathcal{T}(e)$ if and only if $\mathcal{T}(d)$ is finer than $\mathcal{T}(e)$ and $\mathcal{T}(e)$ is finer than $\mathcal{T}(d)$.

We have seen that $\mathcal{T}(d)$ is finer than $\mathcal{T}(e)$ if and only if the identity function on X is $(\mathcal{T}(d), \mathcal{T}(e))$-continuous — or, equivalently, the identity function is (d, e)-continuous. In *Unit A2*, we showed that a function $f\colon X \to X$ is (d, e)-continuous if it is a *Lipschitz function*: that is, if there exists $M > 0$ such that, for all $x, y \in X$,

Theorem 6.1.

Unit A2, Theorem 2.5.

$$e(f(x), f(y)) \leq Md(x, y).$$

So, if there exists $M > 0$ such that $e(x, y) \leq Md(x, y)$ for all $x, y \in X$, then the identity function on X is $(\mathcal{T}(d), \mathcal{T}(e))$-continuous, and so $\mathcal{T}(d)$ is finer than $\mathcal{T}(e)$. We thus have the following result.

Theorem 6.3

Let d and e be metrics on a set X. If there exists $M > 0$ such that

$$e(x, y) \leq Md(x, y), \qquad \text{for all } x, y \in X,$$

then $\mathcal{T}(d)$ is finer than $\mathcal{T}(e)$.

Furthermore, if there exist $K, M > 0$ such that

$$Kd(x, y) \leq e(x, y) \leq Md(x, y), \qquad \text{for all } x, y \in X,$$

then the metrics d and e are topologically equivalent.

This inequality implies that $d(x, y) \leq (1/K)e(x, y)$, and hence $\mathcal{T}(e)$ is finer than $\mathcal{T}(d)$.

The double inequality in Theorem 6.3 tells us that the metrics d and e are metrically equivalent. Thus, the second part of Theorem 6.3 may be rephrased as follows.

Metric equivalence was defined in Subsection 3.3 of *Unit A2*.

Theorem 6.4

Let d and e be metrics on a set X. If d and e are metrically equivalent, then they are topologically equivalent.

Remark

It is important to note that metric equivalence is a *sufficient*, but *not a necessary*, condition for topological equivalence: it is possible for two metric spaces to be topologically equivalent but not metrically equivalent. Nevertheless, showing that two metrics are metrically equivalent is often a convenient way of showing that they are topologically equivalent.

We saw in Theorem 3.3 of *Unit A2* that the product metrics e_1, e_2 and e_∞ are metrically equivalent. We can thus deduce the following result from Theorem 6.4.

Theorem 6.5

The three product metrics e_1, e_2 and e_∞ are topologically equivalent.

Problem 6.3

Let $X = \{a, b, c\}$ and let the metric e on X be defined by

You may assume that e is a metric.

$$e(a,a) = e(b,b) = e(c,c) = 0, \quad e(a,b) = e(b,a) = 2,$$
$$e(b,c) = e(c,b) = 3, \quad e(a,c) = e(c,a) = 4.$$

Show that e is topologically equivalent to the discrete metric d_0.

Solutions to problems

1.1 We must show that $\mathcal{T}$ satisfies (T1)–(T3).

(T1) $\varnothing$ and X are both subsets of X and so belong to $\mathcal{T}$. Thus (T1) is satisfied.

(T2) The intersection of any two subsets of X is also a subset of X and so belongs to $\mathcal{T}$. Thus (T2) is satisfied.

(T3) The union of any collection of subsets of X is also a subset of X and so belongs to $\mathcal{T}$. Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}$ is a topology on X.

1.2 The only subsets of $X = \{a\}$ are $\varnothing$ and X, and so the discrete topology is the same as the indiscrete topology. Since $\varnothing$ and $\{a\}$ belong to *any* topology on $\{a\}$, the collection $\{\varnothing, \{a\}\}$ is indeed the only topology on $\{a\}$.

1.3 In each case, we check whether (T1)–(T3) are satisfied.

(a) (T2) is not satisfied, since

$$\{a,b\} \cap \{a,c\} = \{a\} \notin \mathcal{T}_1.$$

Thus $\mathcal{T}_1$ is not a topology on X.

(b) (T1) is not satisfied, since $\varnothing \notin \mathcal{T}_2$. Thus $\mathcal{T}_2$ is not a topology on X.

(c) (T1) $\varnothing, X \in \mathcal{T}_3$, so (T1) is satisfied.

(T2) The only intersection that we need to check is $\{b\} \cap \{a,b\} = \{b\} \in \mathcal{T}_3$. Thus (T2) is satisfied.

(T3) The only union that we need to check is $\{b\} \cup \{a,b\} = \{a,b\} \in \mathcal{T}_3$. Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}_3$ is a topology on X.

1.4 We begin by noting that

$$f^{-1}(\{r\}) = \varnothing, \quad f^{-1}(\{s\}) = \{a\}, \quad f^{-1}(\{t\}) = \{b,c\}.$$

The topology $\mathcal{T}_Y$ is the same in both cases, and the inverse images of the sets in $\mathcal{T}_Y$ are as follows:

$$\begin{aligned} f^{-1}(\varnothing) &= \varnothing, \\ f^{-1}(\{s\}) &= \{a\}, \\ f^{-1}(\{r,s\}) &= \{a\}, \\ f^{-1}(Y) = f^{-1}(\{r,s,t\}) &= \{a,b,c\} = X. \end{aligned}$$

(a) The inverse images of all the sets in $\mathcal{T}_Y$ belong to $\mathcal{T}_X$, and so f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous.

(b) The inverse images of $\{s\}$ and $\{r,s\}$ do not belong to $\mathcal{T}_X$, and so f is not $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous.

1.5 The only sets in $\mathcal{T}_Y$ are $\varnothing$ and Y. Since $f^{-1}(\varnothing) = \varnothing \in \mathcal{T}_X$ and $f^{-1}(Y) = X \in \mathcal{T}_X$, it follows that f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous.

2.1 The listing of the elements of $\mathbb{Z}$ as

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

leads to the one–one map from $\mathbb{Z}$ to $\mathbb{N}$ defined by

$$f(n) = \begin{cases} 1 & \text{for } n = 0, \\ 2n & \text{for } n > 0, \\ -2n+1 & \text{for } n < 0. \end{cases}$$

(There are other possible maps.) So $\mathbb{Z}$ is countable.

2.2 Suppose that (a,b) is countable, so that there is a one–one map $f\colon (a,b) \to \mathbb{N}$. Consider the map $g : (0,1) \to (a,b)$ given by

$$g(x) = a + (b-a)x.$$

This map is one–one since, if $g(x) = g(y)$, so that $a + (b-a)x = a + (b-a)y$, then $(b-a)x = (b-a)y$; hence, since $a \neq b$, $x = y$. Therefore the map $f \circ g$ is a one–one map from $(0,1)$ to $\mathbb{N}$. But this is impossible, since $(0,1)$ is uncountable (by Theorem 2.2). Hence (a,b) must be uncountable.

2.3 In each case, the correspondence between the index set and the family is one–one.

(a) The index set is $\mathbb{Q}$, which is countably infinite. Thus the family is countably infinite.

(b) The index set is $\{1,2,3,4\}$, which is finite. Thus the family is finite.

(c) The index set is $\mathbb{R}$, which is uncountable. Since there is a one–one correspondence between the index set and elements of the family, we deduce that the family is uncountable.

2.4 We may index the elements of $\mathcal{F}$ by their left endpoints (for example), since any two distinct closed intervals of unit length have distinct left endpoints. Thus

$$\mathcal{F} = \{[q, q+1] : q \in \mathbb{Q}\}$$

displays $\mathcal{F}$ as an indexed family. Since $\mathbb{Q}$ is countably infinite and the correspondence is one–one, $\mathcal{F}$ is countably infinite.

2.5 **(a)** $\bigcup_{i \in I} A_i = \mathbb{Q}$ and $\bigcap_{i \in I} A_i = \varnothing$.

(b) We begin by noting that each set A_i satisfies $A_i \subseteq (-\infty, \infty) = \mathbb{R}$ and so $\bigcup_{i \in I} A_i \subseteq \mathbb{R}$. Also, as i increases in size, so does A_i, and so it seems likely that $\bigcup_{i \in I} A_i = \mathbb{R}$. We now show that this is the case.

If $x \in \mathbb{R}$, then we can always choose $y \in (0, \infty)$ so that $y > |x|$; for example, choose $y = |x| + 1$. Then $x \in A_y$, and so $x \in \bigcup_{i \in I} A_i$. This shows that $\mathbb{R} \subseteq \bigcup_{i \in I} A_i$. Therefore $\bigcup_{i \in I} A_i = \mathbb{R}$.

We now look at the intersection of the sets A_i. We begin by noting that $0 \in A_i$ for each $i \in I$. However, if $x \in \mathbb{R}$ and $x \neq 0$, then, for example, $x \notin A_{|x|/2}$ and so x does not belong to $\bigcap_{i \in I} A_i$. Thus $\bigcap_{i \in I} A_i = \{0\}$.

2.6 Since $f([-1,0]) = [0,1]$ and $f([0,2]) = [0,4]$, $f(A_1) \cap f(A_2) = [0,1]$. However, $A_1 \cap A_2 = \{0\}$, and so $f(A_1 \cap A_2) = \{f(0)\} = \{0\}$, which is a proper subset of $[0,1] = f(A_1) \cap f(A_2)$.

2.7 $A^c = \mathbb{R} - [0,1] = (-\infty, 0) \cup (1, \infty)$.

2.8 **(a)** $A^c = \{2\}$. **(b)** $A^c = \{2,4,5\}$.

2.9 $A_1 = \{2,4,6,8,10,12\}$, $A_2 = \{3,6,9,12\}$;
$A_1 \cap A_2 = \{6,12\}$, $A_1 \cup A_2 = \{2,3,4,6,8,9,10,12\}$.
Thus, $A_1^c = \{1,3,5,7,9,11\}$;
$A_2^c = \{1,2,4,5,7,8,10,11\}$;
$(A_1 \cup A_2)^c = \{1,5,7,11\} = A_1^c \cap A_2^c$;
$(A_1 \cap A_2)^c = \{1,2,3,4,5,7,8,9,10,11\} = A_1^c \cup A_2^c$.

2.10 First law: $(A_1 \cup A_2 \cup A_3)^c = A_1^c \cap A_2^c \cap A_3^c$.
Second law: $(A_1 \cap A_2 \cap A_3)^c = A_1^c \cup A_2^c \cup A_3^c$.

3.1 $\mathbb{R}$ belongs to $\mathcal{T}_1$, by definition. Of the other sets, those that do not contain 1 are $(0,1)$, $(-\infty, 0]$ and $\{2,3,4\}$, and so these sets also belong to $\mathcal{T}_1$.

3.2 $\varnothing$ belongs to the co-finite topology, by definition. Of the other sets, the only one with a finite complement is $\{k \in \mathbb{N} : k \geq 3\}$, since $\{k \in \mathbb{N} : k \geq 3\}^c = \{1,2\}$. So $\{k \in \mathbb{N} : k \geq 3\}$ also belongs to the co-finite topology on $\mathbb{N}$.

3.3 $(\mathbb{R} - \mathbb{Z})^c = \mathbb{Z}$, which is countable. Also, $((-\infty,0) \cup (0,\infty))^c = \{0\}$, which is countable. The other two sets have uncountable complements.
So, only $\mathbb{R} - \mathbb{Z}$ and $(-\infty,0) \cup (0,\infty)$ belong to the co-countable topology on $\mathbb{R}$.

3.4 We must show that $\mathcal{T}$ satisfies (T1)–(T3).

(T1) By definition, $\varnothing \in \mathcal{T}$. Also, $X^c = \varnothing$ is finite and hence countable, and so $X \in \mathcal{T}$. Thus (T1) is satisfied.

(T2) Let $U_1, U_2 \in \mathcal{T}$ and let $U = U_1 \cap U_2$. We must show that $U \in \mathcal{T}$.
If U_1 or U_2 is equal to $\varnothing$, then $U = \varnothing \in \mathcal{T}$. Otherwise, both U_1^c and U_2^c are countable. Thus, by De Morgan's Second Law, $U^c = U_1^c \cup U_2^c$ is the union of two countable sets. By Corollary 2.5, U^c is therefore countable. Hence $U \in \mathcal{T}$. Thus (T2) is satisfied.

(T3) Let $\{U_i : i \in I\}$ be a family of sets in $\mathcal{T}$ and let $U = \bigcup_{i \in I} U_i$. We must show that $U \in \mathcal{T}$.
First suppose that $U_i = \varnothing$ for each $i \in I$; then $U = \varnothing \in \mathcal{T}$. The other possibility is that U_j^c is countable, for some $j \in I$. By De Morgan's First Law, $U^c = \bigcap_{i \in I} U_i^c$. Thus $U^c \subseteq U_j^c$ and so U^c is countable, proving that $U \in \mathcal{T}$. Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}$ is a topology on X.

3.5 Of the listed sets, the only one that does not contain 0 is $(-1,0)$, and the only two that have $(-1,1)$ as a subset are $[-1,1]$ and $(-1,1]$. So, the sets that belong to $\mathcal{T}$ are $(-1,0)$, $[-1,1]$ and $(-1,1]$.

3.6 **(a)** The sets $\{x \in \mathbb{N} : x \geq 10\}$ and $\mathbb{R} - \mathbb{Z}$ are both in $\mathcal{F}_1$ and hence in $\mathcal{T}$. The set $(-\infty,1) \cup (1,\infty)$ is in $\mathcal{F}_2$ and hence in $\mathcal{T}$. The only listed set that is not in $\mathcal{T}$ is $\mathbb{Q}$.

(b) We must show that $\mathcal{T}$ satisfies (T1)–(T3).

(T1) $\varnothing \in \mathcal{F}_1$ and $X \in \mathcal{F}_2$, and so (T1) is satisfied.

(T2) Let $U_1, U_2 \in \mathcal{T}$ and $U = U_1 \cap U_2$. We must show that $U \in \mathcal{T}$.

First, suppose that at least one of the sets U_1 and U_2 (say U_1) is in $\mathcal{F}_1$. Then $U \subseteq U_1$ and so $0 \notin U$, so that $U \in \mathcal{F}_1 \subseteq \mathcal{T}$. The other possibility is that both U_1 and U_2 belong to $\mathcal{F}_2$. In this case, both U_1 and U_2 contain 0 and so U contains 0. Also, by De Morgan's Second Law, $U^c = U_1^c \cup U_2^c$. Since both U_1^c and U_2^c are finite, U^c is also finite. Hence $U \in \mathcal{F}_2 \subseteq \mathcal{T}$. Thus (T2) is satisfied.

(T3) Let $\{U_i : i \in I\}$ be a family of sets in $\mathcal{T}$ and let $U = \bigcup_{i \in I} U_i$. We must show that $U \in \mathcal{T}$.

First, suppose that, for each $i \in I$, $U_i \in \mathcal{F}_1$ and so $0 \notin U_i$. It follows that $0 \notin U$, so that $U \in \mathcal{F}_1 \subseteq \mathcal{T}$. The other possibility is that $U_j \in \mathcal{F}_2$ for some $j \in I$. Thus, for some $j \in I$, $0 \in U_j \subseteq U$. Also, by De Morgan's First Law, $U^c = \bigcap_{i \in I} U_i^c$. Thus $U^c \subseteq U_j^c$ and so U^c is finite. Hence $U \in \mathcal{F}_2 \subseteq \mathcal{T}$. Thus (T3) is satisfied.

Since (T1)–(T3) are satisfied, $\mathcal{T}$ is a topology on X.

3.7 **(a)** Let $U = U_1 \cap U_2$. Then, by De Morgan's Second Law, $U^c = U_1^c \cup U_2^c$. Since U_1^c and U_2^c are both finite, U^c is also finite. Since X is infinite, this implies that U must be infinite and is therefore not empty.

(b) One way is to use a proof by contradiction. Assume that $\mathcal{T} = \mathcal{T}(d)$ for some metric d. Let a and b be distinct points in X and take $r < \frac{1}{2}d(a,b)$. Then $U_1 = B_d(a,r)$ and $U_2 = B_d(b,r)$ are d-open balls and hence belong to $\mathcal{T} = \mathcal{T}(d)$. It follows from (a) that $U_1 \cap U_2 \neq \varnothing$. This, however, is impossible since, if $x \in U_1 \cap U_2$, then, using the triangle inequality,

$$d(a,b) \leq d(a,x) + d(x,b) < 2r < d(a,b).$$

Since our assumption that $\mathcal{T}$ is metrizable has led to a contradiction, it follows that $\mathcal{T}$ is not metrizable.

3.8 Taking the intersection of each of the sets in $\mathcal{T}$ with the set A gives

$$\begin{aligned}\mathcal{T}_A &= \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, \{a,b\}, \{a,b,c\}, A\} \\ &= \{\varnothing, \{a\}, \{a,b\}, \{a,c\}, A\}.\end{aligned}$$

3.9 The sets in $\mathcal{T}_a$ are all the subsets of X that do not contain a, together with X. Since $a \notin A$, taking the intersection of these subsets with A gives all the subsets of A. So the subspace topology on A is the discrete topology on A.

4.1 We have $\mathcal{T}_2 \subseteq \mathcal{T}_1 \subseteq \mathcal{T}_3$. Thus, by Theorem 4.2, the identity function is $(\mathcal{T}_1, \mathcal{T}_2)$-continuous, $(\mathcal{T}_3, \mathcal{T}_1)$-continuous and $(\mathcal{T}_3, \mathcal{T}_2)$-continuous.
(and $(\mathcal{T}_1, \mathcal{T}_1)$-, $(\mathcal{T}_2, \mathcal{T}_2)$-, and $(\mathcal{T}_3, \mathcal{T}_3)$-continuous.)

4.2 If $U \in \mathcal{T}_Y$, then

$$\chi_S^{-1}(U) = \begin{cases} S & \text{if } U = \{1\}, \\ X & \text{if } U = Y, \\ \varnothing & \text{if } U = \varnothing. \end{cases}$$

It follows from the definition of continuity that χ_S is continuous if and only if $\chi_S^{-1}(U) \in \mathcal{T}_X$ in each of these cases. We know that $\varnothing$ and X must belong to $\mathcal{T}_X$, and so f is continuous if and only if S belongs to $\mathcal{T}_X$.

4.3 Let $\mathcal{T}_2$ be the discrete topology on Y and let $\mathcal{T}_1 = \{\varnothing, \{1\}, Y\}$. It follows from Problem 4.2 and Theorem 4.3 that we obtain the specified result if we take a topological space $(X, \mathcal{T}_X)$ and a subset S of X such that S belongs to $\mathcal{T}_X$ but S^c does not. One possibility is to take $X = \{a, b\}$, $\mathcal{T}_X = \{\varnothing, \{a\}, X\}$ and $S = \{a\}$. Then $S \in \mathcal{T}_X$ but $S^c = \{b\} \notin \mathcal{T}_X$.

(There are many possible examples.)

4.4 Let $U \in \mathcal{T}_B$. We must show that $f^{-1}(U) \in \mathcal{T}_X$. By the definition of a subspace topology, there must exist $V \in \mathcal{T}_Y$ with $U = V \cap B$. Now $f(x) \in B$, for any $x \in X$. Thus, $f(x) \in V \cap B = U$ if and only if $f(x) \in V$, and so $f^{-1}(U) = f^{-1}(V)$. Therefore, since f is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous and $V \in \mathcal{T}_Y$, $f^{-1}(U) = f^{-1}(V) \in \mathcal{T}_X$.

Thus $f \colon X \to B$ is $(\mathcal{T}_X, \mathcal{T}_B)$-continuous.

4.5 (a) This function is not one–one since, for example, $f(-1) = f(1) = 1$. Thus it is not a bijection.

(b) This function is not onto since, if $x \in [0, \infty)$, then $f(x) > 0$, and so $f([0, \infty))$ is a proper subset of $\mathbb{R}$. Thus it is not a bijection. (In fact, $f([0, \infty)) = (0, 1]$, and $f \colon [0, \infty) \to (0, 1]$ is a bijection.)

(c) We first show that f is one–one:

$$\begin{aligned} f(x_1) = f(x_2) &\Longrightarrow 2 + 3x_1 = 2 + 3x_2 \\ &\Longrightarrow 3x_1 = 3x_2 \\ &\Longrightarrow x_1 = x_2. \end{aligned}$$

We now show that f is onto. Let $y \in \mathbb{R}$. Then

$$\begin{aligned} f(x) = y &\iff 2 + 3x = y \\ &\iff x = (y - 2)/3. \end{aligned}$$

Since $(y-2)/3 \in \mathbb{R}$, it follows that $y = f((y-2)/3)$ and so f is onto.

Since f is both one–one and onto, it is a bijection.

The inverse function is $f^{-1}(y) = (y - 2)/3$.

4.6 We begin by checking that f is a bijection.

First, we check that f is indeed a mapping from C to E. Let $(x, y) \in C$ and let $f(x, y) = (X, Y)$, so that $X = ax$ and $Y = by$. We must check that $(X/a)^2 + (Y/b)^2 = 1$. This is true because $(X/a)^2 + (Y/b)^2 = x^2 + y^2 = 1$, since $(x, y) \in C$.

Next, we show that f is one–one:

$$\begin{aligned} f(x_1, y_1) = f(x_2, y_2) &\Longrightarrow (ax_1, by_1) = (ax_2, by_2) \\ &\Longrightarrow (x_1, y_1) = (x_2, y_2), \end{aligned}$$

since $a, b \neq 0$.

We now show that f is onto. Let $(X, Y) \in E$. Then

$$\begin{aligned} f(x, y) = (X, Y) &\iff (ax, by) = (X, Y) \\ &\iff (x, y) = (X/a, Y/b). \end{aligned}$$

Since $(X/a)^2 + (Y/b)^2 = 1$, it follows that $(x, y) \in C$. Hence $(X, Y) = f(X/a, Y/b)$ and so f is onto.

Since f is both one–one and onto, it is a bijection.

The inverse function $f^{-1} : E \to C$ is the bijection defined by $f^{-1}(X, Y) = (X/a, Y/b)$.

The functions $x \mapsto ax$, $y \mapsto by$, $X \mapsto X/a$ and $Y \mapsto Y/b$ are basic continuous functions on $\mathbb{R}$ with respect to the Euclidean distance function. Therefore, by Theorem 5.5 of *Unit A1*, $(x, y) \longmapsto (ax, by)$ and $(X, Y) \longmapsto (X/a, Y/b)$ are continuous with respect to the Euclidean distance function on $\mathbb{R}^2$, and hence with respect to the Euclidean topology on $\mathbb{R}^2$. Thus, by Theorems 4.5 and 4.6, f and f^{-1} are continuous with respect to the subspace topologies on C and E inherited from the Euclidean topology on $\mathbb{R}^2$.

Therefore f is a homeomorphism between C and E.

4.7 We must find a bijection from X to Y that gives a one–one correspondence between the sets in $\mathcal{T}_X$ and the sets in $\mathcal{T}_Y$.

One possibility is to define $f \colon X \to Y$ by $f(a) = p$, $f(b) = r$ and $f(c) = q$. This is a bijection, and gives the following one–one correspondence between the open sets:

$$\begin{array}{cccccc} \mathcal{T}_X & \varnothing & \{a\} & \{b\} & \{a,b\} & X \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathcal{T}_Y & \varnothing & \{p\} & \{r\} & \{p,r\} & Y \end{array}$$

(The other possibility is to define $f \colon X \to Y$ by $f(a) = r$, $f(b) = p$ and $f(c) = q$.)

5.1 We must show that (B1) and (B2) are satisfied.

(B1) is immediate, since $\mathcal{C} \subseteq \mathcal{T}$.

(B2) Each open set $U \in \mathcal{T}$ is the union of a family of sets in $\mathcal{B}$, and hence is the union of a family of sets in $\mathcal{C}$. Thus, (B2) is satisfied.

Since (B1) and (B2) are satisfied, $\mathcal{C}$ is a base for $\mathcal{T}$.

5.2 We must show that (B1) and (B2) are satisfied.

(B1) We saw in the remark following Worked problem 4.2 of *Unit A2* that each open square is open with respect to the Euclidean metric on $\mathbb{R}^2$, and therefore belongs to the Euclidean topology on $\mathbb{R}^2$. Thus (B1) is satisfied.

(B2) Let U be open with respect to the Euclidean metric on $\mathbb{R}^2$. We must show that U can be written as a union of open squares. We know from the definition of an open set in a metric space that, for each point $\mathbf{u} \in U$, there is some open ball $B(\mathbf{u})$ of radius $r(\mathbf{u}) > 0$ centred at $\mathbf{u}$ and contained in U. This ball contains the open square $S(\mathbf{u})$ of side $r(\mathbf{u})$ centred at $\mathbf{u}$. Clearly,

$$U \subseteq \bigcup_{\mathbf{u} \in U} S(\mathbf{u}).$$

Since $B(\mathbf{u}) \subseteq U$ for each $\mathbf{u} \in U$, also $S(\mathbf{u}) \subseteq U$ for each $\mathbf{u} \in U$. So

$$\bigcup_{\mathbf{u}\in U} S(\mathbf{u}) \subseteq U.$$

Therefore

$$U = \bigcup_{\mathbf{u}\in U} S(\mathbf{u}).$$

Thus (B2) is satisfied.

Since (B1) and (B2) are satisfied, the result follows.

5.3 We must show that (B1) and (B2) are satisfied.

(B1) If $x \in X$ and $x \neq 0$, then $\{x\} \in \mathcal{F}_1 \subseteq \mathcal{T}$.
Also, $(-1,1) \subseteq (-1,1)$ and so $(-1,1) \in \mathcal{F}_2 \subseteq \mathcal{T}$.
Thus (B1) is satisfied.

(B2) Let $U \in \mathcal{T}$. We must show that U can be written as a union of sets in $\mathcal{B}$.

If $U \in \mathcal{F}_1$, we can write

$$U = \bigcup_{x\in U} \{x\}.$$

Since $0 \notin U$, $\{x\} \in \mathcal{B}$ for each $x \in U$.

If $U \in \mathcal{F}_2$, then U can be written as the union of $(-1,1)$ with either none, one or both of $\{-1\}$ and $\{1\}$. All of these sets belong to $\mathcal{B}$.

Thus (B2) is satisfied.

Since (B1) and (B2) are satisfied, $\mathcal{B}$ is a base for $\mathcal{T}$.

5.4 Let $(a,b) \in \mathcal{B}$. We must show that $f^{-1}((a,b)) \in \mathcal{T}$.

Now $f^{-1}(\{y\}) = \{\sqrt{y}\}$, and hence, since f is an increasing function,

$$f^{-1}((a,b)) = (\sqrt{a}, \sqrt{b}) \in \mathcal{T}.$$

It follows from Theorem 5.3 that f is $(\mathcal{T},\mathcal{T})$-continuous.

5.5 We show that (B3) and (B4) are satisfied.

(B3) We have

$$X = \{a,b,c\} = \varnothing \cup \{a\} \cup \{c\} \cup \{a,b\},$$

and so X is the union of the sets in $\mathcal{B}$.

(B4) We have

$$\{a\} \cap \{c\} = \varnothing \in \mathcal{B},$$
$$\{a\} \cap \{a,b\} = \{a\} \in \mathcal{B},$$
$$\{c\} \cap \{a,b\} = \varnothing \in \mathcal{B}.$$

The intersection of $\varnothing$ with any of the other sets in $\mathcal{B}$ gives $\varnothing$ (which belongs to $\mathcal{B}$). Hence (B4) is satisfied.

Since (B3) and (B4) are satisfied, it follows from Theorem 5.4 that $\mathcal{B}$ is a base for a topology $\mathcal{T}$ on X. The sets in this topology are all the sets that can be formed by taking unions of the sets in $\mathcal{B}$, and so

$$\mathcal{T} = \{\varnothing, \{a\}, \{c\}, \{a,b\}, \{a,c\}, X\}.$$

5.6 Let us write $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$.

We first show that if f is continuous then so are f_1 and f_2. It follows immediately from the Composition Rule (Theorem 4.4) together with Theorem 5.6 that, if f is $(\mathcal{T}_Y,\mathcal{T})$-continuous, then f_1 is $(\mathcal{T}_Y,\mathcal{T}_1)$-continuous and f_2 is $(\mathcal{T}_Y,\mathcal{T}_2)$-continuous.

We now show the converse: if f_1 and f_2 are continuous, then so is f. We use the fact that, by Theorem 5.5,

$$\mathcal{B} = \{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}$$

is a base for the product topology on $X_1 \times X_2$. Therefore, by Theorem 5.3, f is $(\mathcal{T}_Y,\mathcal{T})$-continuous if, for each $U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2$, $f^{-1}(U_1 \times U_2) \in \mathcal{T}_Y$.

Now,

$$\begin{aligned} f_1^{-1}(U_1) &= \{y \in Y : f_1(y) \in U_1\} \\ &= \{y \in Y : (p_1 \circ f)(y) \in U_1\} \\ &= \{y \in Y : f(y) \in U_1 \times X_2\} \\ &= f^{-1}(U_1 \times X_2). \end{aligned}$$

Similarly,

$$\begin{aligned} f_2^{-1}(U_2) &= \{y \in Y : f(y) \in X_1 \times U_2\} \\ &= f^{-1}(X_1 \times U_2). \end{aligned}$$

Hence, using Theorem 2.8,

$$\begin{aligned} f^{-1}(U_1 \times U_2) &= f^{-1}((U_1 \times X_2) \cap (X_1 \times U_2)) \\ &= f^{-1}(U_1 \times X_2) \cap f^{-1}(X_1 \times U_2) \\ &= f_1^{-1}(U_1) \cap f_2^{-1}(U_2). \end{aligned}$$

Now, if f_1 is $(\mathcal{T}_Y, \mathcal{T}_1)$-continuous and f_2 is $(\mathcal{T}_Y, \mathcal{T}_2)$-continuous, then by definition $f_1^{-1}(U_1) \in \mathcal{T}_Y$ and $f_2^{-1}(U_2) \in \mathcal{T}_Y$. Therefore, by (T2),

$$f^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \in \mathcal{T}_Y,$$

and so f is $(\mathcal{T}_Y,\mathcal{T})$-continuous.

6.1 $\mathcal{T}_2$ is finer than $\mathcal{T}_1$ and $\mathcal{T}_3$. $\mathcal{T}_1$ and $\mathcal{T}_3$ are coarser than $\mathcal{T}_2$. $\mathcal{T}_1$ and $\mathcal{T}_3$ are not comparable.

6.2 By the result of Problem 6.1 and Theorem 6.1, the identity function is $(\mathcal{T}_2,\mathcal{T}_1)$-continuous and $(\mathcal{T}_2,\mathcal{T}_3)$-continuous.

6.3 Let $x, y \in X$.

If $x = y$ then $d_0(x,y) = e(x,y) = 0$.

If $x \neq y$ then $d_0(x,y) = 1$ and $2 \leq e(x,y) \leq 4$.

Thus, $2d_0(x,y) \leq e(x,y) \leq 4d_0(x,y)$, for all $x, y \in X$.

Hence, by Theorem 6.3, d_0 and e are topologically equivalent.

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